



A New Optimal Method of Order Four of Hermite–Steffensen Type

I. Păvăloiu and E. Cătiuaş 

Abstract. We introduce a new method for solving nonlinear equations in $\mathbb{R}$, which uses three function evaluations at each step and has convergence order four, being, therefore, optimal in the sense of Kung and Traub. The method is based on the Hermite inverse interpolatory polynomial of degree two. Under certain additional assumptions, we obtain convergence domains (sided intervals) larger than the usual ball convergence sets, and, moreover, with monotone convergence of the iterates. The method has larger convergence domains than of the methods which use intermediate points of the type $y_n = x_n + f(x_n)$ (as the later may not yield convergent iterates when $|f|$ grows fast near the solution).

Mathematics Subject Classification. 65H05.

Keywords. Nonlinear equations in $\mathbb{R}$, inverse interpolation, Hermite–Steffensen type method, computational convergence order.

1. Introduction

In this paper, we introduce and study a Steffensen-type iterative method for solving nonlinear equations in $\mathbb{R}$

$$f(x) = 0,$$

where $f : [a, b] \rightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

Such methods consider an additional equation, equivalent to the above one,

$$g(x) - x = 0$$

with $g : [a, b] \rightarrow [a, b]$.

We assume the following usual hypothesis.

H1) there exists $x^* \in]a, b[$ such that $f(x^*) = 0$ ($\Leftrightarrow g(x^*) = x^*$).

The Steffensen method and its variants use the inverse interpolation polynomial of degree 1 for the function f , and the nodes controlled with the aid of an auxiliary function g [1–3, 5, 9, 11–14, 19].

In this paper we consider a Hermite inverse interpolation polynomial of degree 2, which uses nodes generated with the aid of the function g . We shall obtain an iterative method of order 4, which uses at each step 3 function evaluations, which is, therefore, optimal in the sense of Kung and Traub. Under some reasonable additional assumptions on the function f , we shall show that the obtained iterates converge monotone to the solution.

We shall consider the following hypothesis on the smoothness of the nonlinear mapping:

H2) the function f is three times derivable on $[a, b]$, and $f'(x) \neq 0$, for all $x \in [a, b]$.

This assumption implies that f is monotone and continuous on $[a, b]$, and there exists $f^{-1} : J \rightarrow [a, b]$, $J = f([a, b])$.

By H1), the solution may be written as

$$x^* = f^{-1}(0).$$

Given two approximations $a_1, a_2 \in [a, b]$, of the solution x^* , let $b_1 = f(a_1)$, $b_2 = f(a_2)$ and consider

$$\begin{aligned} a_1 &= f^{-1}(b_1), \\ a_2 &= f^{-1}(b_2), \\ [f^{-1}(y)]'_{y=b_1} &= \frac{1}{f'(a_1)}. \end{aligned}$$

The polynomial of degree 2 which satisfies

$$\begin{aligned} P(b_1) &= a_1, \\ P(b_2) &= a_2, \\ P'(b_1) &= \frac{1}{f'(a_1)} \end{aligned}$$

has the expression given by

$$P(y) = a_1 + [b_1, b_1; f^{-1}](y - b_1) + [b_1, b_1, b_2; f^{-1}](y - b_1)^2$$

with the remainder

$$f^{-1}(y) - P(y) = [y, b_1, b_1, b_2; f^{-1}](y - b_1)^2(y - b_2),$$

where, as usually, $[a_1, a_2; f]$ denotes the divided difference of order one of f at the nodes a_1, a_2 .

Taking $y = 0$ in the above two relations, we get the next approximation by setting $x_3 = P(0)$

$$x^* \simeq P(0) = a_1 - [b_1, b_1; f^{-1}]b_1 + [b_1, b_1, b_2; f^{-1}]b_1^2$$

with the error

$$x^* - P(0) = -[0, b_1, b_1, b_2; f^{-1}]b_1^2b_2.$$

To express the two formulae above in terms of f instead of f^{-1} , we take into account the known relations regarding the divided differences of

the inverse function (see, e.g., [13])

$$\begin{aligned} [b_1, b_1; f^{-1}] &= \frac{1}{f'(a_1)}; \\ [b_1, b_1, b_2; f^{-1}] &= \frac{[a_1, a_1, a_2; f]}{[a_1, a_2, f]^2 f'(a_1)}, \end{aligned}$$

the known mean value theorem (see, e.g., [13]) which yields

$$[0, b_1, b_1, b_2; f^{-1}] = \frac{[f^{-1}(y)]'''_{y=\eta}}{6}, \quad \eta \in \text{int}(J)$$

and the expression for the derivatives of the inverse function (see, e.g., [18])

$$[f^{-1}(y)]''' = \frac{3f''(x)^2 - f'(x)f'''(x)}{f'(x)^5}, \quad y = f(x).$$

Therefore,

$$x^* \simeq a_3 = a_1 - \frac{f(a_1)}{f'(a_1)} - \frac{[a_1, a_1, a_2; f] f^2(a_1)}{[a_1, a_2; f]^2 f'(a_1)} \quad (1)$$

with the approximation error

$$x^* - a_3 = -\frac{E_f(\xi)}{6f'(\xi)^5} f(a_1)^2 f(a_2),$$

where we denoted

$$E_f(x) = 3f''(x)^2 - f'(x)f'''(x).$$

Formula (1) will be used in an iterative fashion, being led to the sequence

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = y_n - \frac{[x_n, x_n, y_n; f] f^2(x_n)}{[x_n, y_n; f]^2 f'(x_n)}, \quad n = 0, 1, \dots \quad (2)$$

As it can be easily seen, each iteration step requires three function evaluations: $f(x_n), f'(x_n), f(y_n)$.

The error can be evaluated by:

$$x^* - x_{n+1} = -\frac{E_f(\xi_n)}{6f'(\xi_n)^5} f^2(x_n) f(y_n). \quad (3)$$

From this formula, we can anticipate that the convergence order of the iterations is 4, i.e., this method is optimal in the sense of Kung and Traub. The efficiency index of the method is $\sqrt[3]{4} \approx 1.587$.

2. Local Convergence and Monotone Convergence

The following local convergence result.

Theorem 1. *Let assumptions H1), H2) hold. If $E_f(x^*) \cdot f''(x^*) \neq 0$, then method (2) converges locally with C-order 4, and the following relations hold:*

$$x^* - x_{n+1} = Q_4(n) (x^* - x_n)^4 \quad (4)$$

with

$$Q_4(n) = \frac{E_f(\xi_n) f''(\delta_n) f'(\mu_n)^2 f'(v_n)}{12 f'(\xi_n)^5 f'(x_n)},$$

where $\xi_n, \delta_n, \mu_n, v_n \in]a, b[$. The asymptotical constant is therefore (see [6] for notations)

$$Q_4 = \lim_{n \rightarrow \infty} Q_4(n) = \frac{E_f(x^*) f''(x^*)}{12 f'(x^*)^3}.$$

If $E_f(x^*) f''(x^*) = 0$, the method converges locally with Q-order at least 4.

Proof. Hypothesis H2) and the first relation in (2) show there exists $\delta_n \in]a, b[$ such that

$$x^* - y_n = -\frac{f''(\delta_n)}{2f'(x_n)} (x^* - x_n)^2. \quad (5)$$

On the other hand,

$$\begin{aligned} f(x_n) &= f'(\mu_n) (x_n - x^*); \\ f(y_n) &= f'(v_n) (y_n - x^*), \quad \mu_n, v_n \in]a, b[. \end{aligned}$$

These relations imply (4) and the other assertions.

To obtain monotone convergence, we shall consider the following additional hypotheses on f :

H3) the expression E_f satisfies

$$E_f(x) = 3f''(x)^2 - f'(x)f'''(x) > 0, \quad \forall x \in [a, b];$$

H4) the initial approximation $x_0 \in [a, b]$ obeys the Fourier condition:

$$f(x_0) f''(x_0) > 0.$$

The following results hold.

Theorem 2. *If the function f and the initial approximation obey the assumptions (H1–H4), and, moreover,*

i₁) $f'(x) > (<)0, \forall x \in [a, b]$;

ii₁) $f''(x) > (<)0, \forall x \in [a, b]$,

then the iterates (2) converge to the solution and are monotone decreasing;

j₁) $x^ < x_{n+1} < y_n < x_n, n = 0, 1, \dots$*

jj₁) $\lim x_n = \lim y_n = x^$.*

Proof. We shall prove by induction (similarly as in [10, 12, 13]). Let $x_n \in [a, b]$ satisfy H4), i.e., $f(x_n) f''(x_n) > 0$, which implies that $f(x_n) > 0$ and, by $i_2)$, $x_n > x^*$. We also have that $y_n < x_n$. Relation (5) and assumptions $i_1)$ and $ii_1)$ show that $x^* - y_n < 0$, i.e., $y_n > x^*$ so $f(y_n) > 0$. By (3) and the hypotheses, it follows that $x^* - x_{n+1} > 0$, i.e., $x_{n+1} > x^*$. The second relation in (2) and the hypotheses imply that $x_{n+1} < y_n$, which prove $j_1)$.

Relation $j_1)$ shows that the sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are convergent: $l = \lim x_n = \lim y_n$. The first relation in (2) implies that $l = l - \frac{f(l)}{f'(l)}$, i.e., $f(l) = 0$, so $l = x^*$.

When the inequalities in $i_1)$ and $ii_1)$ are satisfied with reverse signs, the proof follows in the same way, by taking the function $h = -f$ instead of f . □

Theorem 3. *If the function f and the initial approximation obey the assumptions H1)–H4), and, moreover,*

$$i_2) f'(x) < (>)0, \quad \forall x \in [a, b]$$

$$ii_2) f''(x) > (<)0, \quad \forall x \in [a, b],$$

then the method (2) is monotone increasing:

$$j_2) x_n < y_n < x_{n+1} < x^*, \quad n = 0, 1, \dots$$

$$jj_2) \lim x_n = \lim y_n = x^*$$

The proof is obtained in a similar manner.

3. Numerical Examples

In this section, we shall consider some examples for which we run programs in Julia [4], in quadruple (`binary128` which is implicit for the `BigFloat` type) or higher precision floating point arithmetic (using the `setprecision` command).

There are a lot of optimal iterative methods having the convergence order 4 (see, e.g., the monograph [15]). We shall present only a few of them, for which we have found a better convergence of the above iterates in certain situations.

Example 4. Consider the following equation (see, e.g., [8])

$$f(x) = e^x \sin x + \ln(x^2 + 1), \quad x^* = 0. \quad (6)$$

The largest interval to study the monotone convergence of our method by Theorems 2–3 is $[a, b] := [x^*, 1.54 \dots]$, since f'' vanishes at b (being positive on $[a, b]$). The Fourier condition H4) holds on $[a, b]$ (and does not hold for $x < a$), $E_f(x) > 0$ on $[a, b]$ ($E_f(0) = \frac{46}{3}$), while the derivatives f', f'' are positive on this interval. The conclusions of Theorem 2 apply.

The Hermite–Steffensen iterates (2) lead to the following results, presented in Table 1. Here—as in the other tables—the mantissas are shown truncated.

Table 1. The Hermite–Steffensen iterates (2) for problem (6), quadruple precision

n	x_n	$f(x_n)$	y_n	$f(y_n)$
0	1.54	5.877	5.123324e−1	1.051
1	2.397156e−1	3.576e−1	5.997938e−2	6.723e−2
2	8.721737e−3	8.874e−3	1.474170e−4	1.474e−4
3	8.200791e−8	8.200e−8	1.345059e−14	1.345e−14
4	6.935204e−28	6.935e−28	9.619411e−55	9.619e−55
5	4.660021e−105	4.660e−105	0	0

To verify the C-convergence order, we use the formulae:

$$Q_L(n) = \frac{\ln |x_n - x^*|}{\ln |x_{n-1} - x^*|}, \quad Q'_L(n) = \frac{\ln |x_n - x_{n-1}|}{\ln |x_{n-1} - x_{n-2}|},$$

$$Q_\Lambda(n) = \frac{\ln \frac{|x_n - x^*|}{|x_{n-1} - x^*|}}{\ln \frac{|x_{n-1} - x^*|}{|x_{n-2} - x^*|}}, \quad Q'_\Lambda(n) = \frac{\ln \frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}}{\ln \frac{|x_{n-1} - x_{n-2}|}{|x_{n-2} - x_{n-3}|}},$$

which should converge here to 4 (see, e.g., [6], [7]; here we denote the quantities computable at the step n).

We obtain $Q_L(5) = 3.84$, $Q'_L(5) = 3.83$, $Q_\Lambda(5) = 3.84$, $Q'_\Lambda(5) = 3.99$ ($Q_\Lambda(4) = 3.99$ is better at the previous step).

Using `setprecision(1000)`, we get an additional nonzero iterate x_n , and $Q_L(6) = 3.79$, $Q'_L(6) = 3.95$, $Q_\Lambda(6) = 3.74$, $Q'_\Lambda(6) = 3.99999998$ while $Q_L(5) = 3.95$, $Q'_L(5) = 3.83$, $Q_\Lambda(5) = 3.99999998$, $Q'_\Lambda(5) = 3.993$. The asymptotic constant $Q_4(n)$ is well approximated apart of the last step, when its computed value is $4.8e + 21$ (instead of $\frac{46}{3} = 15.33\dots$).

It is worth noting that the method converges for $-0.2 \leq x_0 < x_1^*$ too (despite the Fourier condition does not hold), as the local convergence near $x_1^* = 0$ is assured by Theorem 1). For $x_0 = -0.3$, the method converges to another solution of the equation, $x_2^* = -0.603\dots$

Now we consider the optimal method of Kanwar et. all in [9, (5.9), $\beta = 1$]

$$y_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n) + f'^2(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n) + f'^2(x_n)} \left[1 + \frac{f(y_{n+1})\{f(x_n) - 2f(y_{n+1})\}\{f^2(x_n) + f'^2(x_n)\}^2}{\{f^2(x_n) + f'^2(x_n)\}^2 \{f(x_n) - 2f(y_{n+1})\}^2 + \alpha f(y_{n+1})f'^2(x_n)} \right],$$

and we took $\alpha = 0.5$, resp. $\alpha = 1$, for which the method has a smaller domain of convergence, as shown in Table 2 (the iterations converge with order four as x_0 is chosen closer to $x^* = 0$). For $\alpha = -1$, resp. $\alpha = 10$ (values also considered by the authors), the iterations converged for $x_0 = 1.54$.

Example 5. Consider the following equation (see, e.g., [14])

$$f(x) = (x - 2)(x^{10} + x + 1)e^{-x-1}, \quad x^* = 2. \quad (7)$$

The largest radius of attraction for the Newton-type iterative methods for computing $x^* = 2$ must be smaller than 0.22, as $f'(1.78\dots) = 0$ (see [13]).

Table 2. The Kanwar et. all iterates (2) for problem (6)

n	x_n ($\alpha = 0.5$)	$f(x_n)$	n	x_n ($\alpha = 1$)	$f(x_n)$
0	1.45	5.3642	0	1.5	5.6491
1	1.4682	5.4682	1	1.5521	5.9474
$\vdots$			$\vdots$		
228	2.4111	9.3560	8	2.2999	9.2766
229	2.4111	9.3560	9	2.3069	9.2869

Table 3. The Hermite–Steffensen iterates (2) for problem (7)

n	$x_n - 2$	$f(x_n)$	$y_n - 2$	$f(y_n)$
0	7.9	761907.13	3.602809	148982.78
1	2.908710	64158.53	2.184591	20149.42
2	1.701263	7456.63	1.264497	2443.69
3	0.947793	906.17	0.657702	298.30
4	0.445481	108.72	0.257942	34.21
5	1.323053e-1	11.23	4.334529e-2	2.628
6	7.861441e-3	4.147e-1	2.377742e-4	1.216e-2
7	3.481418e-7	1.780e-5	4.831580e-13	2.470e-11
8	1.467014e-24	7.501e-23	8.579185e-48	4.386e-46
9	4.625388e-94	2.365e-92	0	0

The largest interval to study the monotone convergence of our method by Theorems 2–3 is $[a, b] := [x^*, 7.9 \dots]$, since f'' vanishes at b (being positive on $[a, b]$).

The Fourier condition H4) holds on $[a, b]$ (and does not hold for $x < a$), $E_f(x) > 0$ on $[a, b]$, while both the derivatives f', f'' are positive on this interval. The conclusions of Theorem 2 apply.

The iterates (2) leads to the results presented in Table 3 (`setprecision(500)` was used instead of the implicit quadruple precision). The iterates converge even for $x_0 > 7.9$ (and to the left of the solution as well, but for x_0 higher than 1.72). Of course, the convergence may be not very fast when the initial approximations are away from the solution.

For the convergence orders, we obtain $Q_L(9) = 3.91$, $Q'_L(9) = 3.69$, $Q_\Lambda(9) = 3.9999998$, $Q'_\Lambda(9) = 3.990$.

Let us consider the optimal method of Ren, Wu, Bi [16]

$$\begin{aligned}
 z_n &= x_n + f(x_n), \\
 y_n &= x_n - \frac{f(x_n)}{[x_n, z_n; f]}, \\
 x_{n+1} &= y_n - \frac{f(y_n)}{[x_n, y_n; f] + [y_n, z_n; f] - [x_n, z_n; f] + a(y_n - x_n)(y_n - z_n)}, \quad (8)
 \end{aligned}$$

Table 4. The Ren–Wu–Bi iterates (8) for problem (7)

n	x_n ($a = 1$)	$f(x_n)$	x_n ($a = -1$)	$f(x_n)$
0	2.3	45.8747	2.3	45.8747
1	48.1539	1.3906e-3	48.1975	1.3447e-3
2	49.4519	5.0943e-4	49.4957	4.9239e-4
3	50.7395	1.8669e-4	50.7832	1.8042e-4
4	52.0177	6.8443e-5	52.0611	6.6140e-5

Table 5. The Liu–Zheng–Zhao iterates (9) for problem (7)

n	x_n	$f(x_n)$
0	2.3	45.8747
1	48.1788	1.3642e-3
2	50.6609	1.9854e-4
3	53.1081	2.8922e-5
4	55.5250	4.2161e-6

for the parameter a having two given values: 1, -1 (when $a = 0$ the iterates behave similarly). The convergence domain is smaller, as the iterates do not converge to $x^* = 2$ for $x_0 = 2.3$, as it can be seen in Table 4. The explanation resides in the fact that the values of z_n may increase with the values of f .

Next we consider the optimal method of Zhongli Liu, Quan Zheng, Peng Zhao [19]

$$\begin{aligned}
 z_n &= x_n + f(x_n), \\
 y_n &= x_n - \frac{f(x_n)}{[x_n, z_n; f]}, \\
 x_{n+1} &= y_n - \frac{[x_n, y_n; f] - [y_n, z_n; f] + [x_n, z_n; f]}{[x_n, y_n; f]^2} f(y_n).
 \end{aligned} \tag{9}$$

This method also has a small convergence domain here, as the iterates do not converge to $x^* = 2$ for $x_0 = 2.3$ (see Table 5).

We end with some interesting results given by the optimal method of Sharma–Guha [17]:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 4f(y_n)/f(x_n)}} \frac{f(x_n)}{f'(x_n)}. \tag{10}$$

This method has a pretty large domain of convergence here (we took $x^* = 7.9$ and beyond), but the iterates away from the solution ($x_0 \geq 2.2$) are complex numbers. We illustrate some results in Table 6.

Conclusions. The method studied in this paper present, under certain circumstances, some advantages over other optimal methods (specifically, for those which use intermediate points of the type $y_n = x_n + f(x_n)$). The obtained sufficient conditions for guaranteed convergence may theoretically

Table 6. The Sharma–Guha iterates (10) for problem (7)

n	x_n	$f(x_n)$	n	x_n	$f(x_n)$
0	7.9	761907.13	0	2.2	21.6789
1	$4.76 + 0.0i$	$52513.99 + 0.0i$	1	$1.98 - 6.12e-2i$	$-1.50 - 2.65i$
2	$3.68 - 0.558i$	$2544.21 - 8056.48i$	2	$1.99 + 3.25e-4i$	$-1.42e-2 + 1.66e-2i$
...			3	$2.00 - 2.78e-13i$	$4.38e-11 - 1.42e-11i$
9	$1.99 - 3.76e-26$	$-1.52e-25 - 1.92e-24$	4	$1.99 + 1.68e-47i$	$-2.84e-46 + 8.64e-46i$
10	$2.0 + 1.78e-101i$	$-6.50e-200 + 9.13e-100i$	5	$2.0 + 3.78e-124$	$-2.91e-245 + 1.93e-122$
11	$2.0 + 0.0i$	$0.0 + 0.0i$	6	$2.0 + 0.0i$	$2.0 + 0.0i$

lead to larger convergence domains (especially sided convergence intervals) than from estimates of attraction balls, while certain examples shown that the attraction domain of the method is larger than for some optimal methods.

The test problems presented seem to be suitable for testing the iterative methods on their convergence domains.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Amat, S., Blanda, J., Busquier, S.: A Steffensen type method with modified functions. *Riv. Mat. Univ. Parma* **7**, 125–133 (2007)
- [2] Argyros, I.K.: *A new convergence theorem for the Steffensen method in Banach space and applications*, *Rev. Anal. Numér. Théor. Approx.*, 29 (2000) no. 1, 119–127. accessed online: <https://ictp.acad.ro/jnaat/journal/article/view/2000-vol29-no2-art1> on May 1st, 2017
- [3] Argyros, I.K.: Á. Alberto Magreñán, *On the convergence of an optimal fourth-order family of methods and its dynamics*. *Appl. Math. Comput.* **252**, 336–346 (2015). <https://doi.org/10.1016/j.amc.2014.11.074>
- [4] Bezanson, J., Edelman, A., Karpinski, S., Shah, V.B.: Julia: A fresh approach to numerical computing. *SIAM Rev.* **59**, 65–98 (2017). <https://doi.org/10.1137/141000671>
- [5] Cătiņaș, E.: *On some Steffensen-type iterative method for a class of nonlinear equations*, *Rev. Anal. Numér. Théor. Approx.*, **24** (1995) no. 1-2, 37–43. accessed online: <https://ictp.acad.ro/jnaat/journal/article/view/1994-vol23-no1-art4> on May 1st, 2017
- [6] Cătiņaș, E.: A survey on the high convergence orders and computational convergence orders of sequences. *Appl. Math. Comput.* **343**, 1–20 (2019). <https://doi.org/10.1016/j.amc.2018.08.006>
- [7] Cătiņaș, E.: How many steps still left to x^* ? *SIAM Rev.* **63**(3), 585–624 (2021). <https://doi.org/10.1137/19M1244858>
- [8] Chun, C.: Certain improvements of Chebyshev-Halley methods with accelerated fourth-order convergence. *Appl. Math. Comput.* **189**(1), 597–601 (2007). <https://doi.org/10.1016/j.amc.2006.11.118>

- [9] Kanwar, V.: Ramandeep Behl, Kapil K. Sharma.: *Simply constructed family of a Ostrowski's method with optimal order of convergence*. Comput. Math. Appl. **62**, 4021–4027 (2011). <https://doi.org/10.1016/j.camwa.2011.09.039>
- [10] Păvăloiu, I.: Approximation of the root of equations by Aitken-Steffensen-type monotonic sequences. Calcolo **32**(1–2), 69–82 (1995). <https://doi.org/10.1007/bf02576543>
- [11] Păvăloiu, I., Cătiņaș, E.: Bilateral approximation for some Aitken-Steffensen-Hermite type method of order three. Appl. Math. Comput. **217**(12), 5838–5846 (2011). <https://doi.org/10.1016/j.amc.2010.12.067>
- [12] Păvăloiu, I., Cătiņaș, E.: On an Aitken-Newton type method. Numer. Algor. **62**(2), 253–260 (2013). <https://doi.org/10.1007/s11075-012-9577-7>
- [13] Păvăloiu, I., Cătiņaș, E.: On a robust Aitken-Newton method based on the Hermite polynomial. Appl. Math. Comput. **287–288**, 224–231 (2016). <https://doi.org/10.1016/j.amc.2016.03.036>
- [14] Petković, M.S.: On a general class of multipoint root-finding methods of high computational efficiency. SIAM J. Numer. Anal. **47**(6), 4402–4414 (2010). <https://doi.org/10.1137/090758763>
- [15] Petković, M.S., Neta, B., Petković, L.D., Džunić, J.: Multipoint Methods For Solving Nonlinear Equations. Academic Press, New York (2013)
- [16] Ren, H., Wu, Q., Bi, W.: A class of two-step Steffensen type methods with fourth-order convergence. Appl. Math. Comput. **209**, 206–210 (2009). <https://doi.org/10.1016/j.amc.2008.12.039>
- [17] Sharma, J.R., Guha, R.K.: Second-derivative free methods of third and fourth order for solving nonlinear equations. Int. J. Comput. Math. **88**(1), 163–170 (2011). <https://doi.org/10.1080/00207160903365875>
- [18] Turowicz, B.A.: Sur le dérivées d'ordre supérieur d'une fonction inverse. Ann. Polon. Math. **8**, 265–269 (1960)
- [19] Liu, Zhongli, Zheng, Quan, Zhao, Peng.: A variant of Steffensen's method of fourth-order convergence and its applications. Appl. Math. Comput. **216**, 1978–1983 (2010). <https://doi.org/10.1016/j.amc.2010.03.028>

I. Păvăloiu and E. Cătiņaș

“Tiberiu Popoviciu” Institute of Numerical Analysis, Romanian Academy

P.O. Box 68-1

Cluj-Napoca

Romania

e-mail: ecatinas@ictp.acad.ro

Received: February 22, 2021.

Revised: July 30, 2021.

Accepted: April 5, 2022.