

Article

On Ulam Stability of a Partial Differential Operator in Banach Spaces

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Abstract: In this paper, we prove that, if $\inf_{x \in A} |f(x)| = m > 0$, then the partial differential operator

D defined by $D(u) = \sum_{k=1}^n f_k \frac{\partial u}{\partial x_k} - fu$, where $f, f_i \in C(A, \mathbb{R})$, $u \in C^1(A, X)$, $i = 1, \dots, n$, $I \subset \mathbb{R}$ is an

interval, $A = I \times \mathbb{R}^{n-1}$ and X is a Banach space, is Ulam stable with the Ulam constant $K = \frac{1}{m}$. Moreover, if $\inf_{x \in A} |f(x)| = 0$, we prove that D is not generally Ulam stable.

Keywords: Ulam stability; partial differential operator; gauge; Banach space

MSC: 39B82; 47A50



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1. Introduction

Ulam stability is one of the main topics in the theory of functional equations. Generally, a functional equation is called Ulam stable (or Hyers–Ulam stable) if for every approximate solution of the equation, there exists a solution of the equation near it. The problem of stability of functional equations was formulated by S. M. Ulam in a talk given at the University of Wisconsin–Madison and concerns the equations of group homomorphism [1]. The first answer to Ulam’s problem was given by D. H. Hyers who proved that Cauchy’s functional equation in Banach spaces is stable [2]. For this reason, the stability of a functional equation is called Ulam stability or Hyers–Ulam stability. After Hyers’ result was published, a lot of papers were dedicated to the study of Ulam stability, and the notion of Ulam stability was extended in various directions (for more detail, see the books [3,4]). In recent years, many papers have been published dedicated to the study of Ulam stability for differential equations and differential operators. For example, the authors of [5] studies the Ulam stability of a second order linear differential operator acting in a Banach space. They obtained a characterization of its Ulam stability and the best Ulam constant. Similar results were obtained by the authors of [6] for a first order linear differential equation with periodic coefficients. For various kinds of operational equations, including partial and ordinary differential equations, results regarding their Ulam stability are presented in [7]. Recent results on the Ulam stability of partial differential equations of order one are given in [8,9]. The stability of a parabolic partial differential equation is presented in [10] and an explicit form of the Ulam constant for the Laplace operator is obtained in [3] (see Section 3). It seems that A. Prastaro and Th. M. Rassias published the first paper on the Ulam stability of a partial differential equation [11]. The relation between a solution of a perturbed equation and its exact solution is established by the Ulam stability of an equation or of an operator. If small perturbations of the equation produce small perturbations of the solution, we say that the equation is Ulam stable. Ulam stability of an operator means Ulam stability of its associated equation. The goal of this paper is to present a result on the Ulam stability of a linear and nonhomogeneous partial differential operator acting in Banach spaces and

to obtain an explicit form of its Ulam constant. We improve and extend in this way some results for partial differential equations with constant coefficients given in [8] and for partial differential equations with nonconstant coefficients with two variables given in [9].

Let $I = [a, b]$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $A = I \times \mathbb{R}^{n-1}$, X be a Banach space over $\mathbb{R}$ and V be a vector space over $\mathbb{R}$.

Throughout this paper, by $\|\cdot\|$ we denote the norm of the Banach space X and by $\|\cdot\|_e$ the euclidian norm in $\mathbb{R}^k$.

Definition 1. A function $\rho_V : V \rightarrow [0, +\infty]$ is called a gauge on V if the following properties hold:

- (i) $\rho_V(x) = 0 \Leftrightarrow x = 0$;
- (ii) $\rho_V(\lambda x) = |\lambda| \rho_V(x)$, $\forall x \in V, \forall \lambda \in \mathbb{R}, \lambda \neq 0$.

Let $f, f_1, f_2, \dots, f_n : A \rightarrow \mathbb{R}$ be continuous functions and $D : C^1(A, X) \rightarrow C(A, X)$ be defined by

$$D(u) = \sum_{k=1}^n f_k \frac{\partial u}{\partial x_k} - fu, \quad u \in C^1(A, X). \quad (1)$$

Let $\varphi \in C(A, X)$ and define

$$\|\varphi\|_\infty = \sup\{\|\varphi(x)\| : x \in A\}. \quad (2)$$

Then, $\|\varphi\|_\infty$ is a gauge on $C(A, X)$. Consider the same gauge on $C^1(A, X)$.

Definition 2. The operator D is called Ulam stable if there exists $K \geq 0$ such that, for every $\varepsilon > 0$ and every $u \in C^1(A, X)$ with the property

$$\|D(u)\|_\infty \leq \varepsilon, \quad (3)$$

there exists $u_0 \in C^1(A, X)$ such that $D(u_0) = 0$ and

$$\|u - u_0\|_\infty \leq K\varepsilon. \quad (4)$$

The number K is called an Ulam constant of D .

A function $u \in C^1(A, X)$ satisfying (3) for some positive ε is called an approximate solution of the equation $D(u) = 0$. Thus, we can reformulate the definition of D as follows: the operator D is said to be Ulam stable if for every approximate solution u of the associated equation $D(u) = 0$, there exists an element $u_0 \in \ker D$ (i.e., a solution of $D(u) = 0$) close to u (i.e., satisfying (4)).

The Ulam stability of some linear operator acting in spaces endowed with gauges has been studied in [3]. Since gauges are generalized norms or metrics, results on the characterization and on the best Ulam constant of such operators extend the results on Ulam stability given in normed spaces for linear operators (see [3], Sections 2 and 3). A detailed presentation of Ulam stability for some operators relative to gauges is given in [12]. Here, the authors obtain Ulam stability characterization theorems and results regarding the representation of the best Ulam constant. Concrete examples are obtained for differential operators. Results related to the Ulam stability of partial differential operators are not found in [12]. Our goal is to give such results in the present paper.

2. Main Result

The following result contains a representation of the solutions of the equation $D(u) = g$.

Lemma 1. Let $g \in C(A)$ and suppose that $f_1 \neq 0$ on A . Let $g_k = \frac{f_k}{f_1}$, $2 \leq k \leq n$. Suppose that the system

$$\begin{cases} x'_2 = g_2(t, x_2, \dots, x_n) \\ \vdots \\ x'_n = g_n(t, x_2, \dots, x_n) \end{cases}, t \in I \quad (5)$$

admits a global solution $(\varphi_2, \dots, \varphi_n) : I \rightarrow \mathbb{R}^{n-1}$. Then, u is a solution of the equation

$$f_1(x) \frac{\partial u}{\partial x_1} + \dots + f_n(x) \frac{\partial u}{\partial x_n} - f(x)u = g(x), x = (x_1, \dots, x_n) \in A \quad (6)$$

if and only if there exists a function $F \in C^1(\mathbb{R}^{n-1}, X)$ such that

$$\begin{aligned} u(x_1, \dots, x_n) &= e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \\ &\cdot \left(\int_a^{x_1} \frac{e^{L(s, x_2 - \varphi_2(s), \dots, x_n - \varphi_n(s))} g(s, x_2 - \varphi_2(s) + \varphi_2(s), \dots, x_n - \varphi_n(s) + \varphi_n(s))}{f_1(s, x_2 - \varphi_2(s) + \varphi_2(s), \dots, x_n - \varphi_n(s) + \varphi_n(s))} ds \right. \\ &\quad \left. + F(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)) \right), \end{aligned} \quad (7)$$

where $x = (x_1, \dots, x_n) \in A$, and

$$L(x_1, x_2, \dots, x_n) = - \int_a^{x_1} \frac{f(s, x_2 + \varphi_2(s), \dots, x_n + \varphi_n(s))}{f_1(s, x_2 + \varphi_2(s), \dots, x_n + \varphi_n(s))} ds.$$

Proof. Suppose that u satisfies (6) and consider the change of coordinates:

$$\begin{cases} t_1 = x_1 \\ t_2 = x_2 - \varphi_2(x_1) \\ \vdots \\ t_n = x_n - \varphi_n(x_1) \end{cases} \Leftrightarrow \begin{cases} x_1 = t_1 \\ x_2 = t_2 + \varphi_2(t_1) \\ \vdots \\ x_n = t_n + \varphi_n(t_1) \end{cases}. \quad (8)$$

Let

$$\gamma(t) = (t_1, t_2 + \varphi_2(t_1), \dots, t_n + \varphi_n(t_1)), t = (t_1, \dots, t_n).$$

Define the function w by the relations

$$w(t_1, t_2, \dots, t_n) = u(t_1, t_2 + \varphi_2(t_1), \dots, t_n + \varphi_n(t_1)) = u(\gamma(t))$$

or

$$u(x_1, \dots, x_n) = w(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)).$$

Then,

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= \frac{\partial w}{\partial t_1} - \frac{\partial w}{\partial t_2} \varphi'_2(t_1) - \dots - \frac{\partial w}{\partial t_n} \varphi'_n(t_1) \\ \frac{\partial u}{\partial x_k} &= \frac{\partial w}{\partial t_k}, k = 2, 3, \dots, n. \end{aligned}$$

Replacing this in (6), we obtain

$$\begin{aligned} f_1(\gamma(t)) \left(\frac{\partial w}{\partial t_1} - \varphi'_2(t_1) \frac{\partial w}{\partial t_2} - \dots - \varphi'_n(t_1) \frac{\partial w}{\partial t_n} \right) &+ f_2(\gamma(t)) \frac{\partial w}{\partial t_2} + \dots + f_n(\gamma(t)) \frac{\partial w}{\partial t_n} - f(\gamma(t))w \\ &= g(\gamma(t)), \end{aligned}$$

or

$$f_1(\gamma(t)) \frac{\partial w}{\partial t_1} + (f_2(\gamma(t)) - f_1(\gamma(t))\varphi'_2(t_1)) \frac{\partial w}{\partial t_2} + \dots + (f_n(\gamma(t)) - f_1(\gamma(t))\varphi'_n(t_1)) \frac{\partial w}{\partial t_n} - f(\gamma(t))w = g(\gamma(t)).$$

Now, taking account of (5), i.e.,

$$\varphi'_k = g_k = \frac{f_k}{f_1}, \quad 2 \leq k \leq n,$$

it follows that

$$\frac{\partial w}{\partial t_1} - \frac{f(\gamma(t))}{f_1(\gamma(t))} w = \frac{g(\gamma(t))}{f_1(\gamma(t))}, \quad t = (t_1, \dots, t_n) \in I \times \mathbb{R}^{n-1}. \quad (9)$$

Let the function L be defined by

$$L(t_1, t_2, \dots, t_n) = - \int_a^{t_1} \frac{f(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} ds, \quad (t_1, t_2, \dots, t_n) \in I \times \mathbb{R}^{n-1}.$$

Then, multiplying Equation (9) by $e^{L(t_1, t_2, \dots, t_n)}$, we obtain

$$\frac{\partial}{\partial t_1} \left(w(t_1, t_2, \dots, t_n) \cdot e^{L(t_1, t_2, \dots, t_n)} \right) = \frac{g(t_1, t_2 + \varphi_2(t_1), \dots, t_n + \varphi_n(t_1))}{f_1(t_1, t_2 + \varphi_2(t_1), \dots, t_n + \varphi_n(t_1))} e^{L(t_1, t_2, \dots, t_n)},$$

which leads to

$$w(t_1, t_2, \dots, t_n) \cdot e^{L(t_1, t_2, \dots, t_n)} = \int_a^{t_1} \frac{g(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s)) e^{L(s, t_2, \dots, t_n)}}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} ds + F(t_2, \dots, t_n),$$

where F is an arbitrary function of class C^1 . Then,

$$w(t_1, t_2, \dots, t_n) = e^{-L(t_1, t_2, \dots, t_n)} \left(\int_a^{t_1} \frac{g(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s)) e^{L(s, t_2, \dots, t_n)}}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} ds + F(t_2, \dots, t_n) \right) \quad (10)$$

Replacing $t_1, t_2, \dots, t_n$ from (8) into (10), the relation (7) is obtained.

Now, let u be given by (7), and we must prove that u is a solution of (1). Taking account of the change of coordinates (8), it is sufficient to prove that w given by (10) satisfies (9). A simple calculation shows that w is a solution of (9). $\square$

The main result of this paper is given in the next theorem.

Theorem 1. Let $\varepsilon > 0$ be a given number, $f_1 \neq 0$ on A and suppose that:

- (i) The system (5) admits a global solution $(\varphi_2, \dots, \varphi_n) : I \rightarrow \mathbb{R}^{n-1}$;
- (ii) $\inf_{x \in A} |f(x)| = m > 0$.

Then, for every $u \in C^1(A, X)$ satisfying $\|D(u)\|_\infty \leq \varepsilon$, there exists a solution u_0 of $D(u) = 0$ with the property

$$\|u - u_0\|_\infty \leq \frac{\varepsilon}{m}. \quad (11)$$

Moreover, if $L(b, x_2, \dots, x_n) := \lim_{x_1 \rightarrow b} L(x) = -\infty$, $\forall (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, then u_0 is uniquely determined.

Proof. Since f_1 is continuous on the connected set A and $f_1 \neq 0$, it follows that $f_1(x) > 0$ or $f_1(x) < 0$ for every $x \in A$. We may suppose in what follows that $f_1(x) > 0$ for every $x \in A$, without loss of generality.

Existence. Let u be a solution of (3) and put

$$\sum_{k=1}^n f_k(x) \frac{\partial u(x)}{\partial x_k} - f(x)u(x) := g(x)$$

for every $x = (x_1, \dots, x_n) \in A$. Then, according to Lemma 1, we have

$$\begin{aligned} & u(x_1, \dots, x_n) \\ &= e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \\ & \cdot \left(\int_a^{x_1} \frac{e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} g(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds \right. \\ & \left. + F(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)) \right) \end{aligned}$$

where $F \in C^1(\mathbb{R}^{n-1}, X)$.

We consider the function u_0 defined by

$$\begin{aligned} & u_0(x_1, \dots, x_n) = e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \\ & \cdot \left(\int_a^b \frac{e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} g(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds \right. \\ & \left. + F(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)) \right) \end{aligned}$$

for every $x = (x_1, \dots, x_n) \in A$. Since u_0 is given by an integral on the noncompact interval $[a, b]$, first we have to prove that the function u_0 is well defined, i.e., the improper integral is convergent.

We test the convergence of the integral

$$\int_a^b \frac{e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} g(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds,$$

where $x = (x_1, \dots, x_n) \in A$ or, according to (8), the convergence of the following integral

$$\int_a^b \frac{e^{L(s, t_2, \dots, t_n)} g(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} ds. \quad (12)$$

We have

$$\begin{aligned} & \left\| e^{L(s, t_2, \dots, t_n)} \frac{g(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} ds \right\| \\ & \leq \frac{e^{L(s, t_2, \dots, t_n)}}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} \|g(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))\| \\ & \leq \frac{e^{L(s, t_2, \dots, t_n)}}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} \varepsilon \\ & = \frac{\varepsilon e^{L(s, t_2, \dots, t_n)}}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} \cdot \frac{f(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))}{f_1(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} \\ & \leq \frac{\varepsilon}{m} \cdot \frac{\partial}{\partial s} \left(e^{-L(s, t_2, \dots, t_n)} \right), \quad (s, t_2, \dots, t_n) \in I \times \mathbb{R}^{n-1}. \end{aligned}$$

Since

$$\int_a^b \frac{\varepsilon}{m} \frac{\partial}{\partial s} \left(e^{-L(s, t_2, \dots, t_n)} \right) ds = \frac{\varepsilon}{m} \left(1 - e^{L(b, t_2, \dots, t_n)} \right) \leq \frac{\varepsilon}{m},$$

it follows, in view of the comparison test, that the integral (12) is absolutely convergent; therefore, the function u_0 is well defined. On the other hand, u_0 is a solution of (1) being of form (7).

We have:

$$\begin{aligned}
 & \|u(x) - u_0(x)\| \\
 &= \left\| e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \int_{x_1}^b -e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \right. \\
 &\quad \cdot \left. \frac{g(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds \right\| \\
 &\leq e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \int_{x_1}^b \frac{\varepsilon e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))}}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds \\
 &\leq e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \int_{x_1}^b \frac{\varepsilon e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))}}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} \\
 &\quad \cdot \frac{f(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))}{f(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s))} ds \\
 &\leq \frac{\varepsilon}{m} e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \int_{x_1}^b \frac{f(s, t_2 + \varphi_2(s), \dots, t_n + \varphi_n(s)) e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))}}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds \\
 &\leq \frac{\varepsilon}{m} e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \int_{x_1}^b \frac{\partial}{\partial s} \left(-e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \right) ds \\
 &= \frac{\varepsilon}{m} e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \left(e^{L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} - e^{L(b, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \right) \\
 &= \frac{\varepsilon}{m} \left(1 - e^{L(b, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)) - L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \right) \leq \frac{\varepsilon}{m}, \quad \forall (x_1, \dots, x_n) \in A,
 \end{aligned}$$

so

$$\|u - u_0\|_\infty \leq \frac{\varepsilon}{m}.$$

The existence is proved.

Uniqueness.

Suppose that $L(b, x_2, \dots, x_n) = -\infty$ and for a solution u of (3) there exists two solutions u_1, u_2 of $D(u) = 0$, $u_1 \neq u_2$ with the property (11). Then, $u_k, k = 1, 2$, are given by

$$\begin{aligned}
 & u_k(x_1, \dots, x_n) \\
 &= e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \\
 &\quad \cdot \left(\int_a^{x_1} \frac{f(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s)) e^{L(s, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))}}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds \right. \\
 &\quad \left. + F_k(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)) \right), \quad k = 1, 2, \quad F_1 \neq F_2.
 \end{aligned}$$

We have

$$\begin{aligned}
 & \|u_1(x_1, \dots, x_n) - u_2(x_1, \dots, x_n)\| \\
 &= \|u_1(x_1, \dots, x_n) - u(x_1, \dots, x_n)\| + \|u(x_1, \dots, x_n) - u_2(x_1, \dots, x_n)\| \\
 &\leq \frac{2\varepsilon}{m}, \quad \forall (x_1, \dots, x_n) \in A,
 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
 & e^{-L(x_1, x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))} \|F_1(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)) - F_2(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1))\| \\
 &\leq \frac{2\varepsilon}{m}, \quad \forall (x_1, \dots, x_n) \in A.
 \end{aligned} \tag{13}$$

Since $u_1 \neq u_2$, it follows that there exists $(\tilde{x}_2, \dots, \tilde{x}_n) \in \mathbb{R}^{n-1}$ such that $F_1(\tilde{x}_2, \dots, \tilde{x}_n) \neq F_2(\tilde{x}_2, \dots, \tilde{x}_n)$. For

$$\begin{cases} x_2 - \varphi_2(x_1) =: \tilde{x}_2 \\ \vdots \\ x_n - \varphi_n(x_1) =: \tilde{x}_n \end{cases}$$

the relation (13) becomes

$$e^{-L(x_1, \tilde{x}_2, \dots, \tilde{x}_n)} \|F_1(\tilde{x}_2, \dots, \tilde{x}_n) - F_2(\tilde{x}_2, \dots, \tilde{x}_n)\| \leq \frac{2\varepsilon}{m}, \forall x_1 \in I. \quad (14)$$

Now, letting $x_1 \rightarrow b$ in (14), it follows $\infty \leq \frac{2\varepsilon}{m}$, a contradiction. Uniqueness is proven. $\square$

A consequence of the stability result given in Theorem 1 is presented in the next corollary.

Corollary 1. Let $F = (f_2, \dots, f_n) : A \rightarrow \mathbb{R}^{n-1}$, $f_1 \neq 0$ on A . Suppose that there exists two continuous functions $\alpha, \beta : I \rightarrow [0, +\infty)$ such that

$$\|F(t, x)\|_e \leq (\alpha(t)\|x\|_e + \beta(t))|f_1(t, x)| \quad (15)$$

for every $(t, x) \in I \times \mathbb{R}^{n-1}$ and $\inf_{x \in A} |f(x)| = m$, $m > 0$. Then, the operator D given by (1) is Ulam stable with the Ulam constant $L = \frac{1}{m}$.

Proof. Let $G = (g_2, \dots, g_n)$, $g_k = \frac{f_k}{f_1}$, $2 \leq k \leq n$. Then, by (15) we obtain

$$\|G(t, x)\|_e \leq \alpha(t)\|x\|_e + \beta(t), \quad (t, x) \in A,$$

and taking account of Theorem 2.4.5 ([13], see also [14]), we conclude that system (5) admits a global solution $\varphi : I \rightarrow \mathbb{R}^{n-1}$. Now, the conclusion of the corollary is a consequence of Theorem 1. $\square$

Remark 1. The conclusion of Corollary 1 holds true if on the right hand side of relation (15) f_1 is replaced by f_p , $1 \leq p \leq n$, with $f_p \neq 0$ on A .

Example 1. Let $a_1, \dots, a_n \in \mathbb{R}$, $a_1 \neq 0$, $A = [0, +\infty) \times \mathbb{R}^{n-1}$, $f : A \rightarrow \mathbb{R}$ be a continuous function with $\inf_{x \in A} |f(x)| = m$, $m > 0$. Then, the operator with constant coefficients $D : C^1(A, X) \rightarrow C(A, X)$

$$D(u) = \sum_{k=1}^n a_k \frac{\partial u}{\partial x_k} - fu, \quad u \in C^1(A, X)$$

is Ulam stable with the Ulam constant $L = \frac{1}{m}$.

Proof. Clearly the system (5) associated with the operator D has a global solution $\varphi : [0, +\infty) \rightarrow \mathbb{R}^{n-1}$, so the conclusion follows according to Theorem 1. $\square$

Example 2. Let $A = I \times \mathbb{R}$, $f_1, f_2, f \in C(A, \mathbb{R})$ and $D : C^1(A, X) \rightarrow C(A, X)$,

$$D(u) = f_1 \frac{\partial u}{\partial x_1} + f_2 \frac{\partial u}{\partial x_2} - fu.$$

If $\inf_{(x_1, x_2) \in A} |f(x_1, x_2)| = m, m > 0$ and there exists two continuous functions $\alpha, \beta : I \rightarrow [0, \infty)$ such that

$$\left| \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \right| \leq \alpha(x_1)|x_2| + \beta(x_2), \quad (16)$$

for all $(x_1, x_2) \in A$, then D is Ulam stable with the Ulam constant $L = \frac{1}{m}$.

Proof. The condition (16) leads to the existence of a global solution of the system (5), which in this case is the equation

$$x_2' = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.$$

The conclusion follows in view of Corollary 1. $\square$

3. A Nonstability Result

We show in what follows that if the condition $\inf_{x \in A} |f(x)| = m, m > 0$ is not satisfied, then the operator D defined by (1) is not generally Ulam stable. In this respect, we have the following nonstability result for the case $f = 0$.

Theorem 2. Let $I = [a, \infty)$, $A = I \times \mathbb{R}^{n-1}, n \geq 2$. Suppose that the system (5) admits a solution $(\varphi_2, \dots, \varphi_n) : I \rightarrow \mathbb{R}^{n-1}$ and there exists $p \in \{1, \dots, n\}$ such that $f_p \neq 0$ on A and $\sup_{x \in A} |f_p(x)| < +\infty$. Then, the operator $\tilde{D} : C^1(A, X) \rightarrow C(A, X)$

$$\tilde{D}(u) = \sum_{k=1}^n f_k \frac{\partial u}{\partial x_k}, \quad u \in C^1(A, X)$$

is not Ulam stable.

Proof. Without loss of generality, we may suppose that $p = 1$ and let $\sup |f_1(x)| = \lambda, \lambda \neq \infty$. Suppose that $\tilde{D}$ is Ulam stable with the Ulam constant $L > 0$. Let $\varepsilon > 0$ and $u \in C^1(A, X)$ be a solution of the equation $\tilde{D}(u) = \varepsilon \theta, \theta \in X, \|\theta\| = 1$. Then, according to Lemma 1, u has the form

$$u(x) = \int_a^{x_1} \frac{\varepsilon \theta}{f_1(s, x_2 - \varphi_2(x_1) + \varphi_2(s), \dots, x_n - \varphi_n(x_1) + \varphi_n(s))} ds + F(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)),$$

where $F \in C^1(\mathbb{R}^{n-1}, X), x = (x_1, \dots, x_n) \in A$.

Since $\|\tilde{D}(u)\|_\infty = \varepsilon$, it follows, in view of the stability of $\tilde{D}$, that there exists $u_0 \in \ker \tilde{D}$ such that

$$\|u - u_0\|_\infty \leq L\varepsilon. \quad (17)$$

Now, taking into account again Lemma 1, we obtain

$$u_0(x) = G(x_2 - \varphi_2(x_1), \dots, x_n - \varphi_n(x_1)), \quad x = (x_1, \dots, x_n) \in A,$$

for some $G \in C^1(\mathbb{R}^{n-1}, X)$.

Let $z(x) = u(x) - w(x), x \in A$ and let $x_k = \varphi_k(x_1), 2 \leq k \leq n$.

Then,

$$z(x_1, \varphi_2(x_1), \dots, \varphi_n(x_1)) = \int_a^{x_1} \frac{\varepsilon \theta}{f_1(s, \varphi_2(s), \dots, \varphi_n(s))} ds + F(0, \dots, 0) - G(0, \dots, 0),$$

and

$$\begin{aligned} & \|z(x_1, \varphi_2(x_1), \dots, \varphi_n(x_1)) - F(0, \dots, 0) + G(0, \dots, 0)\| \\ &= \int_a^{x_1} \frac{\varepsilon}{|f_1(s, \varphi_2(s), \dots, \varphi_n(s))|} ds \geq \\ &\geq \frac{\varepsilon}{\lambda}(x_1 - a), \forall x_1 \in [a, \infty). \end{aligned}$$

Therefore, denoting $F(0, \dots, 0) - G(0, \dots, 0) = h$, $h \in X$, we obtain

$$\begin{aligned} \|z(x_1, \varphi_2(x_1), \dots, \varphi_n(x_1))\| &\geq \|z(x_1, \varphi_2(x_1), \dots, \varphi_n(x_1)) - h\| - \|h\| \\ &\geq \frac{\varepsilon}{\lambda}(x_1 - a) - \|h\|, \forall x_1 \in [a, \infty). \end{aligned}$$

Hence,

$$\lim_{x_1 \rightarrow \infty} \|z(x_1, \varphi_2(x_1), \dots, \varphi_n(x_1))\| = +\infty,$$

a contradiction to (17). The theorem is proven. $\square$

Example 3. Let $a_1, a_2, \dots, a_n \in \mathbb{R}$, $(a_1, \dots, a_n) \neq (0, \dots, 0)$, $I = [a, \infty)$, $A = I \times \mathbb{R}^{n-1}$, and $D : C^1(A, X) \rightarrow C(A, X)$

$$D(u) = \sum_{k=1}^n a_k \frac{\partial u}{\partial x_k}, \quad u \in C^1(A, X).$$

Then, D is not Ulam stable.

Proof. Suppose $a_1 \neq 0$ without loss of generality. Then, system (5) admits in this case a global solution and the operator D is not Ulam stable according to Theorem 2. $\square$

4. An Application

One of the best-known forms for the study of production functions in economics is the Cobb–Douglas production function (see [15]). C.W. Cobb, a mathematician, and P.H. Douglas, an economist, obtained a function which provides a relation between labor, capital and the production function. Cobb proposed a relation of the form

$$Q = AL^\alpha K^\beta \quad (18)$$

where Q is the production function, L is the labor, K is the capital and A, α, β are some positive constants. Note that Q is a homogeneous function of degree $\alpha + \beta$, so it satisfies Euler's partial differential equation

$$L \frac{\partial Q}{\partial L} + K \frac{\partial Q}{\partial K} = (\alpha + \beta)Q. \quad (19)$$

Over time, Cobb–Douglas production functions have received appreciation and criticism from many scientists (see [16]). That is why we think that the consideration of the notion of an approximate production function is justified. We call an approximate production function a function which approximately satisfies Equation (19) in the sense defined in this paper. In this respect, the study of the Ulam stability of Equation (19) is important. We have the following result.

Theorem 3. Let $\varepsilon > 0$ and suppose that an approximate production function $Q(L, K)$ satisfies the relation

$$\left| L \frac{\partial Q}{\partial L} + K \frac{\partial Q}{\partial K} - (\alpha + \beta)Q \right| \leq \varepsilon,$$

for all $L, K > 0$. Then, there exists a production function $\bar{Q}(L, K)$, i.e., $\bar{Q}$ satisfies Equation (19), such that

$$|Q(L, K) - \bar{Q}(L, K)| \leq \frac{\varepsilon}{\alpha + \beta},$$

for all $L, K > 0$.

Proof. The proof follows as a simple consequence of Theorem 1. $\square$

We conclude that the Cobb–Douglas equation is Ulam stable, so small perturbations of it produce small perturbations of the production function.

Some other practical applications of Ulam stability for the partial differential operator D can be obtained for some processes governed by first order semilinear partial differential equations, such as conservation laws, traffic flow, linear transport equations, etc.

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