

NUMERICAL STABILITY OF THE SPLINE COLLOCATION METHODS FOR VOLTERRA INTEGRAL EQUATIONS

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1. INTRODUCTION

Recently, we have presented in [2] and [3] a method for the construction of an approximation to the solution of nonlinear Volterra integral equation of the second kind. The exact solution y of the integral equation is approximated by an element u of the polynomial spline space $S_{m+d}^{(d)}(Z_N)$, whose elements are the spline functions of degree $m+d$ in continuity class C^d , (m and d are two integers with $m \geq 1$ and $d \geq -1$). This approximation u is determined by collocation methods. The attainable order of global convergence and local superconvergence of these methods is analyzed in detail. In this paper we will investigate the numerical stability of these methods in the case when the mesh sequence $\{\Pi_N\}$ is uniform, for various values of m and d , and for various sets of collocation parameters.

The polynomial spline collocation method is described in the next section. In section 3 a stability criteria is presented, which is given in terms of the eigenvalues of a characteristic matrix. Using this result, some particular cases are discussed in section 4.

2. THE SPLINE COLLOCATION METHOD

We consider the nonlinear Volterra integral equation of the second kind:

$$y(t) = f(t) + \int_0^t K(t, s, y(s)) ds, \quad t \in I := [0, T], \quad (2.1)$$

with the given functions $f : I \rightarrow R$ and $K : S \times R \rightarrow R$ with $S := \{(t, s) : 0 \leq s \leq t \leq T\}$, which are supposed to be sufficiently smooth so that the integral equation (2.1) would have a unique solution $y \in C^\alpha(I)$, with $\alpha \in N$.

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In order to describe this method let $\Pi_N : 0 = t_0 < t_1 < \dots < t_N = T$ (with $t_n = t_n^{(N)}$) be a quasi-uniform mesh for the given interval I , and set:

$$\begin{aligned}\sigma_0 &:= [t_0, t_1], \sigma_n := (t_n, t_{n+1}], \quad n = 1, 2, \dots, N-1, \\ h_n &:= t_{n+1} - t_n, h = \max_{(n)} h_n, \quad n = 0, 1, 2, \dots, N-1, \\ Z_N &:= \{t_n : n = 1, \dots, N-1\}, \overline{Z_N} = Z_N \cup \{T\}.\end{aligned}$$

Moreover, let $\mathcal{P}_k$ denote the space of (real) polynomials of degree not exceeding k . We then define, for given integers m and d with $m \geq 1$ and $d \geq -1$,

$$\begin{aligned}S_{m+d}^{(d)}(Z_N) &:= \{u : u(t) \mid_{t \in \sigma_n} = u_n(t) \in \mathcal{P}_{m+d}, n = 0, \dots, N-1, \\ u_{n-1}^{(j)}(t_n) &= u_n^{(j)}(t_n) \text{ for } j = 0, 1, \dots, d \text{ and } t_n \in Z_N\},\end{aligned}$$

to be the space of polynomial splines of degree $m+d$, whose elements possess the knots Z_N and are d times continually differentiable on I . If $d = -1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinuity at the knots Z_N .

An element $u \in S_{m+d}^{(d)}(Z_N)$ has for all $n = 0, \dots, N-1$, and for all $t \in \sigma_n$, the following form (see [6]):

$$u(t) := u_n(t) = \sum_{r=0}^d \frac{u_{n-1}^{(r)}(t_n)}{r!} (t - t_n)^r + \sum_{r=1}^m a_{n,r} (t - t_n)^{d+r}. \quad (2.2)$$

From (2.2) we see that an element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined when we know the coefficients $\{a_{n,r}\}_{r=1, \dots, m}$ for all $n = 0, \dots, N-1$. In order to determine these coefficients we consider the set of collocation parameters $\{c_j\}_{j=1, \dots, m}$, where $0 < c_1 < \dots < c_m \leq 1$, and we define the set of collocation points by:

$$X(N) := \bigcup_{n=0}^{N-1} X_n, \text{ with } X_n := \{t_{n,j} := t_n + c_j h_n, j = 1, 2, \dots, m\}.$$

The approximate solution $u \in S_{m+d}^{(d)}(Z_N)$ will be determined imposing the condition that u satisfies the Volterra integral equation (2.1) on $X(N)$:

$$u(t) = f(t) + \int_0^t K(t, s, u(s)) ds, \text{ for all } t \in X(N). \quad (2.3)$$

For every choice of the collocation parameters $\{c_j\}_{j=1, \dots, m}$ with $0 < c_1 < c_2 < \dots < c_m \leq 1$ and for all quasi-uniform mesh sequences $\{\Pi_N\}$ with sufficiently small $h > 0$, the above algorithm determines a unique approximate solution $u \in S_{m+d}^{(d)}(Z_N)$, (see [2] and [3]).

In this paper, we will analyze the numerical stability of the polynomial spline collocation method in the case in which the mesh sequences $\{\Pi_N\}$ are uniform, i.e. $h_n = h$, for all $n = 0, 1, \dots, N-1$.

3. NUMERICAL STABILITY

In order to discuss numerical stability we study the behavior of the method as applied to the integral equation:

$$y(t) = f(t) + \lambda \int_0^t y(s) ds, \quad t \in I := [0, T], \quad (3.1)$$

λ being a constant with negative real part, and f a smooth given function. This equation is known in the literature as the basic test equation. This test equation was proposed by Mayers in [5], and it has been used in the study of various numerical methods. In particular, El Tom study the numerical stability of the polynomial spline collocation method in the case when the mesh sequences $\{\Pi_N\}$ is uniform, and the collocation parameters are $c_j := \frac{j}{m}$, for $j = 1, \dots, m$ (see [4]).

Henceforward, we shall refer to a polynomial spline collocation method in the space $S_{m+d}^{(d)}(Z_N)$, simply as an (m, d) -method.

Definition 3.1. An (m, d) -method is said to be stable if all solutions $\{u(t_n)\}$ remain bounded, as $n \rightarrow \infty$, $h \rightarrow 0$ while hN remains fixed.

It is convenient at this stage to introduce some more notations. We write for $n = 0, 1, \dots, N-1$:

$$\eta_n := (\eta_{n,r})_{r=0, \dots, d}^T, \text{ where } \eta_{n,r} := \frac{u_{n-1}^{(r)}(t_n)}{r!} h^r, \quad (3.2)$$

and

$$\beta_n := (\beta_{n,r})_{r=1, \dots, m}^T, \text{ where } \beta_{n,r} := a_{n,r} h^{d+r}. \quad (3.3)$$

With this notation an element $u \in S_{m+d}^{(d)}(Z_N)$ has for all $n = 0, \dots, N-1$ and for any $t = t_n + \tau h_n \in \sigma_n$, the following form:

$$u_n(t + \tau h_n) = \sum_{r=0}^d \eta_{n,r} \tau^r + \sum_{r=1}^m \beta_{n,r} \tau^{d+r}, \text{ for } \tau \in (0, 1]. \quad (3.4)$$

From (3.2) and (3.3) we observe that the set of coefficients η_n is determined by the smooth conditions and the coefficients β_n are determined by collocation equations. Moreover, for $n = 0$, the values of η_0 are determined by the values of the exact solution y and its derivatives in the initial point $t = 0$.

Now, if we apply the collocation method to test integral equation (3.1) and we use the representation (3.4) we obtain the following collocation equation:

$$\begin{aligned}\sum_{r=1}^m \left(1 - \frac{\lambda h c_j}{d+r+1}\right) c_j^{d+r} \beta_{n,r} &= \sum_{r=0}^d \left(\frac{\lambda h c_j}{r+1} - 1\right) c_j^r \eta_{n,r} + f(t_{n,j}) + \\ &+ \sum_{i=0}^{n-1} h \lambda \int_0^1 u_i(t_i + \tau h) d\tau,\end{aligned} \quad (3.5)$$

for $j = 1, \dots, m$ ($n = 0, \dots, N-1$).

In the case when $d \geq 0$, from collocation equation (3.5) we obtain a relation between η_n and β_n :

$$V\beta_n = W\eta_n + r_n, \quad n = 0, 1, \dots, N-1, \quad (3.6)$$

where V is the $m \times m$ matrix, W is the $m \times (d+1)$ matrix and r_n is the m -vector whose elements are:

$$\begin{aligned} v_{j,r} &:= \left(1 - \frac{\lambda h c_j}{d+r+1}\right) c_j^{d+r}; \\ w_{j,r} &:= \begin{cases} \lambda h c_j, & \text{for } r = 0, j = 1, \dots, m, \\ -\left(1 - \frac{\lambda h c_j}{d+r+1}\right) c_j^r, & \text{for } r = 1, \dots, d, j = 1, \dots, m; \end{cases} \\ r_{0,j} &:= f(t_{0,j}) - f(t_0), \end{aligned}$$

and for $n > 0$

$$r_{n,j} := f(t_{n,j}) - f(t_{n-1,m}) + u_{n-1}(t_{n-1,m}) - u_{n-1}(t_n) + \lambda h \int_{c_m}^1 u_{n-1}(t_{n-1} + \tau h) d\tau.$$

By direct differentiation of relations (3.4), for the smooth conditions we get a relation between vector η_{n+1} and vectors η_n and β_n respectively:

$$\eta_{n+1} = A\eta_n + B\beta_n, \text{ for } n = 0, 1, \dots, N-2, \quad (3.7)$$

where A is the $(d+1) \times (d+1)$ upper triangular matrix and B is the $(d+1) \times m$ matrix whose elements are:

$$a_{j,r} := \begin{cases} 0 & , \text{ if } r < j \\ \binom{r}{j} & , \text{ if } r \geq j \end{cases}, \quad b_{j,r} := \binom{d+r}{j}.$$

In order to prove the results concerning the numerical stability properties of the polynomial spline collocation method we need the following lemmas (see [7]):

Lemma 3.2. For any matrix P and any $\varepsilon > 0$, there exists some subordinate norm such that $\|P\| \leq S(P) + \varepsilon$, with $S(P) := \max\{|\lambda_j| : \lambda_j \text{ are the eigenvalues of } P\}$. If P is of class $\mathbf{M}$, then there exists a norm such that $\|P\| = S(P)$.

By means of this lemma we can characterize the numerical stability in the terms of the eigenvalues of suitable matrix. The following theorem represents a stability criteria for our method.

Theorem 3.3. An (m, d) -method is stable if and only if all eigenvalues of matrix $M := A + BV^{-1}W$ are in the unit disk, and all eigenvalues with $|\mu| = 1$ belong to 1×1 Jordan block.

Proof. The aim of this proof consists in showing that the vectors η_n and β_n defined by (3.2) and (3.3) are uniformly bounded for all n as $h \rightarrow 0$ and kN remains fixed. This in turn implies, according to (3.4), that $u(t_n)$ are bounded, as $h \rightarrow 0$, $n \rightarrow \infty$ while kN remains fixed.

From the form of matrix V we see that for h small enough this matrix is non singular. Elimination of β_n between (3.6) and (3.7) yields:

$$\eta_{n+1} = M\eta_n + BV^{-1}r_n, \text{ with } M := A + BV^{-1}W, \quad (3.8)$$

for all $n = 0, \dots, N-2$. Thus, the relations (3.8) and (3.6) imply that for all $n = 0, \dots, N-1$ we have:

$$\begin{aligned} \eta_n &= M^n \eta_0 + \sum_{i=0}^{n-1} M^{n-1-i} BV^{-1} r_i, \\ \beta_n &= V^{-1} W \left[M^n \eta_0 + \sum_{i=0}^{n-1} M^{n-1-i} BV^{-1} r_i \right] + V^{-1} r_n. \end{aligned} \quad (3.9)$$

Because the first derivative of the given function f is a continuous function on I , it results that there exists a positive constant L such that $|f'(t)| \leq L$, for all $t \in I$ and, for all $n = 0, \dots, N-1$ we have:

$$\begin{aligned} \|r_n\|_1 &:= \sum_{j=1}^m |r_{n,j}| \leq \sum_{j=1}^m [Lh(1 - c_m + c_j) + \\ &+ |u_{n-1}(t_{n-1,m}) - u_{n-1}(t_n)| + |\lambda| h \int_{c_m}^1 |u_{n-1}(t_{n-1} + \tau h)| d\tau]. \end{aligned} \quad (3.10)$$

In the case in which $c_m = 1$ relation (3.10) becomes:

$$\|r_n\|_1 \leq hL \sum_{j=1}^m c_j \leq hL_1, \text{ with } L_1 := L \sum_{j=1}^m c_j \leq mL, \quad (3.11)$$

and from relation (3.9) we obtain:

$$\begin{aligned} \|\eta_n\|_1 &\leq \|M\|^n \|\eta_0\|_1 + hL_1 \|BV^{-1}\|_1 \sum_{i=0}^{n-1} \|M\|^{n-1-i}, \\ \|\beta_n\|_1 &\leq \|V^{-1}W\|_1 \cdot \|\eta_n\|_1 + h \|V^{-1}\|_1 L_1, \quad n = 0, 1, \dots, N-1. \end{aligned} \quad (3.12)$$

Using Lemma 3.2, it results from (3.12) that:

$$\begin{aligned} \|\eta_n\| &\leq (S(M))^n \|\eta_0\| + hL_1 \|BV^{-1}\| \sum_{i=0}^{n-1} (S(M))^{n-1-i}, \\ \|\beta_n\| &\leq \|V^{-1}W\| \|\eta_n\| + h \|V^{-1}\| L_1, \quad (n = 0, \dots, N-1). \end{aligned} \quad (3.13)$$

From these relations it results that $\|\eta_n\|_1$ and $\|\beta_n\|_1$ remain bounded for $n \rightarrow \infty$, $h \rightarrow 0$ and $Nh = T$ if and only if $S(M) \leq 1$.

If $c_m \neq 1$ then we will prove by induction that relations (3.12) and (3.13) holds if we change the constant L_1 with a new finite constant L_2 defined by:

$$\begin{aligned} L_2 &:= \max\{L_{2,n} : n = 0, \dots, N-1\}, \text{ with } L_{2,0} := L_1 \text{ and} \\ L_{2,n} &:= L \sum_{j=1}^m (1 - c_m + c_j) + m(1 - c_m) (M_n^1 + |\lambda| M_n), \text{ for } n \geq 1, \end{aligned} \quad (3.14)$$

where $M_n = \max\{|u_{n-1}(t)| : t \in \sigma_{n-1}\}$ and $M_n^1 = \max\{|u'_{n-1}(t)| : t \in \sigma_{n-1}\}$.

For $n = 0$ relations (3.9) become: $\eta_0 = \eta_0$ and $\beta_0 = V^{-1}W\eta_0 + V^{-1}r_0$, respectively. Because the matrices V^{-1} , W and the vector r_0 are bounded for $h \rightarrow 0$, it results that the vector β_0 is bounded too. Thus by the definition relation (3.4) we obtain $|u_0(\tau h)| < \infty$ and $|u'_0(\tau h)| < \infty$, for all $\tau \in [0, 1]$, hence, by (3.10) it results that $\|r_1\|_1 \leq hL_{2,1}$ with $L_{2,1} < \infty$.

Now we suppose that $\|\beta_j\|_1 \leq \infty$ and $\|\eta_j\|_1 < \infty$ for all $j = 0, 1, \dots, n-1$. Under this assumption, by (3.4) it results that $|u_{n-1}(t)| < \infty$ and $|u'_{n-1}(t)| < \infty$ for all $t \in \sigma_{n-1}$; furthermore, by (3.10) it follows that $\|r_n\|_1 \leq hL_{2,n}$, with $L_{2,n} < \infty$. Moreover, relations (3.7) and (3.6) imply $\|\eta_n\| < \infty$, and $\|\beta_n\| < \infty$, respectively. Thus, using the bounded of r_n , from (3.9) it results that the relations (3.12) and (3.13) hold with L_1 replaced by L_2 , for all $n = 0, 1, \dots, N-1$; accordingly the theorem is fully demonstrated.

4. APPLICATIONS

In the following we will investigate some special cases.

I. $d = 0$. In this case the approximation space is $S_m^{(0)}(Z_N)$. An element $u \in S_m^{(0)}(Z_N)$ has for all $n = 0, \dots, N-1$, the form:

$$u_n(t_n + \tau h) = u_{n-1}(t_n) + \sum_{r=1}^m \beta_{n,r} \tau^r, \text{ for } \tau \in (0, 1]. \quad (4.1)$$

If we denote by u_{n+1} the vector with m -elements $u_{n+1} := (u_n(t_n + c_j h))_{j=1, \dots, m}^T$, then from equation (4.1) we obtain:

$$u_{n+1} = (1, 1, \dots, 1)^T u_{n-1}(t_n) + E \cdot \beta_n, \text{ for } n = 0, \dots, N-1, \quad (4.2)$$

with the matrix E defined by $E := (c_j^r)_{j,r=1, \dots, m}$.

In this case the collocation equation becomes:

$$V\beta_n = Wu_{n-1}(t_n) + r_n, \text{ for all } n = 0, \dots, N-1, \quad (4.3)$$

with the matrix W defined by $W := (\lambda h c_j^r)_{j,r=1, \dots, m}$.

Because $V = E + O(h)$ and $W = O(\lambda h)$, the elimination of β_n between (4.2) and (4.3) yields:

$$u_{n+1} = [(1, \dots, 1)^T + O(\lambda h) + O(\lambda^2 h^2)] u_{n-1}(t_n) + (I + O(\lambda h)) r_n. \quad (4.4)$$

If $c_m = 1$, then the point $t_{n-1,m} := t_{n-1} + c_m h = t_n \in Z_N$, and by equation (4.4) it results:

$$u(t_{n+1}) = (1 + O(\lambda h) + O(\lambda^2 h^2))^{n+1} u_{-1}(0) + \sum_{i=0}^n (1 + O(\lambda h) + O(\lambda^2 h^2))^{n-i} (1 + O(\lambda h)) \cdot r_{i,m} \quad (4.5)$$

Since $u_{-1}(0) := y(0) = f(0)$, from (4.5) and (3.11), for $n \rightarrow \infty$ ($h \searrow 0, Nh = T$) it follows:

$$|u(t_{n+1})| \leq |f(0)| + L_1 T < \infty, \text{ for all } n = 0, \dots, N-1. \quad (4.6)$$

If $c_m \neq 1$ ($t_{n-1,m} \neq t_n$), we introduce the constant $c_0 := 0$, and define $t_{n,0} := t_n + c_0 h$. Then, for all $\tau \in [0, 1]$, the approximation $u \in S_m^{(0)}(Z_N)$ may be written:

$$u_n(t_n + \tau h) = \sum_{j=0}^m L_j(\tau) u_n(t_{n,j}), \text{ with } u_n(t_{n,0}) := u_{n-1}(t_n), \quad (4.7)$$

where:

$$L_j(\tau) := \prod_{\substack{i=0 \\ i \neq j}}^m \frac{(\tau - c_i)}{(c_j - c_i)}$$

are the Lagrange fundamental polynomial associated with the collocation parameters $\{c_j\}_{j=1, \dots, m}$. Now substituting $u_n(t_{n,j})$ in (4.7) with its values given by (4.4), for all $n = 0, 1, \dots, N-1$, we obtain:

$$u_n(t_n + \tau h) = \left[1 + (O(\lambda h) + O(\lambda^2 h^2)) \sum_{j=1}^m L_j(\tau) \right] u_{n-1}(t_n) + (1 + O(\lambda h)) \sum_{j=0}^m L_j(\tau) r_{n,j}, \text{ for } \tau \in [0, 1]. \quad (4.8)$$

Because $\sum_{j=1}^m |L_j(\tau)| =: Q < \infty$ for all $\tau \in [0, 1]$, then for $\tau = 1$, equations (4.8) imply that for all $n = 0, \dots, N-1$, we have:

$$|u_n(t_{n+1})| \leq (1 + O(\lambda h) + O(\lambda^2 h^2))^{n+1} y(0) + Q \sum_{i=0}^n (1 + O(\lambda h) + O(\lambda^2 h^2))^{n-i} (1 + O(\lambda h)) \|r_i\|_1 \quad (4.9)$$

Like in proof of Theorem 3.3, we may prove by induction that for all $n = 0, \dots, N-1$, $\|r_n\|_1 \leq hL_2 < \infty$, with L_2 defined by formula (3.14), and thus from (4.9) it results:

$$|u(t_{n+1})| \leq |y(0)| + QL_2 T \text{ for all } n = 0, 1, \dots, N-1, \quad (4.10)$$

Thus, from relations (4.6) and (4.10) it results the following stability theorem.

Theorem 4.1. An $(m, 0)$ -method is stable for all $m \geq 1$ and for every choice of the collocation parameters $\{c_j\}_{j=1, \dots, m}$.

II. $m = 1$. This choice of m corresponds to a classical spline function, i.e. $u \in S_{d+1}^{(d)}(Z_N)$, $d \geq 0$. We denote by M_0 the matrix M with $h = 0$, and by μ^0 and μ the eigenvalues of M_0 and M respectively. It then follows that:

$$\mu = \mu^0 + O(h),$$

If $c_1 \in (0, 1]$ is the collocation parameter then, for all $d \geq 0$, using the binomial expansion we find the trace:

$$\text{Tr}(M_0) = d + 2 + \frac{1}{c_1^{d+1}} - \left(1 + \frac{1}{c_1}\right)^{d+1} \quad (4.11)$$

Concerning the stability of the spline collocation method we have the following result.

Theorem 4.2. An $(1, d)$ -method is stable if and only if one from the following conditions is true:

- (i) $d = 0$ and $c_1 \in (0, 1]$;
- (ii) $d = 1$ and $c_1 = 1$.

Proof. From the form of matrix M_0 we observe that $\mu^0 = 1$ is an eigenvalue of this matrix for all $d \geq 0$ and $c_1 \in (0, 1]$. Because the matrix M_0 is a square matrix with $\dim M_0 = (d+1)$, it results that in the case $d = 0$, $\mu^0 = 1$ is the single eigenvalue of M_0 , and thus an $(1, 0)$ -method is stable for all $c_1 \in (0, 1]$. If $d = 1$ then the second eigenvalue of M_0 is $\mu_2^0 = 1 - \frac{2}{c_1} \leq -1$ for $c_1 \in (0, 1]$, and its absolute value is 1 if and only if $c_1 = 1$. For $d \geq 2$, from relation (4.11) we obtain:

$$-\infty < \text{Tr}(M_0) < -(d+1), \text{ for } d > 2 \text{ and } c_1 \in (0, 1],$$

and $\mu_2^0 + \mu_3^0 \leq -4$, for $d = 2$. Since $\text{Tr}(M_0) = \mu_1^0 + \mu_2^0 + \dots + \mu_{d+1}^{(c)} < -(d+1)$ it results that there exists an eigenvalue μ^0 whose value is smaller than -1 , i.e. $|\mu^0| > 1$. Thus, for $d \geq 2$ an $(1, d)$ -method is unstable for every choice of the collocation parameter $c_1 \in (0, 1]$.

III. $m = 2$. In this case we find for the trace of matrix M_0 the relation:

$$\text{Tr}(M_0) = d + 3 + \frac{(1+c_2)^{d+1}(c_1-c_2-1)+(1-c_1)}{c_2^{d+1}(c_2-c_1)} - \frac{(1+c_1)^{d+1}(c_2-c_1-1)+(1-c_2)}{c_1^{d+1}(c_2-c_1)}, \quad (4.12)$$

where $0 < c_1 < c_2 \leq 1$ are the collocation parameters. Using the above relation we obtain the following theorem.

Theorem 4.3. (i) An $(2, 0)$ -method is stable for every choice of the collocation parameters.

(ii) An $(2, 1)$ -method is stable if and only if $c_1 + c_2 \geq \frac{3}{2}$.

(iii) If $c_2 = 1$, then an $(2, d)$ -method is unstable for all $d \geq 2$.

Proof. Assertion (i) follows from Theorem 4.1 or from the fact that $\mu_1^0 = 1$ is an eigenvalue of the matrix M_0 for all $m \geq 1$ and $d > 0$. To prove the assertion (ii), it is enough to observe that, for $d = 1$, the second eigenvalue of M_0 is $\mu_2^0 = \frac{c_1 c_2 - 2(c_1 + c_2) + 3}{c_1 c_2}$, and the stability condition $|\mu_2^0| \leq 1$ is equivalent with the condition $c_1 + c_2 \geq \frac{3}{2}$.

If $d = 2$ and $c_2 = 1$, then one of the eigenvalues of M_0 is:

$$\mu_2^0 = \frac{1}{2c_1^2} \left(-4c_1^2 + 4c_1 + 1 + \sqrt{12c_1^4 - 24c_1^3 + 4c_1^2 + 8c_1 + 1} \right);$$

here we have $\mu_2^0 > 1$ for every choice of the collocation parameter $c_1 \in (0, 1)$. Thus assertion (iii) holds for $d = 2$. If $d > 2$, then for $c_2 = 1$ the formula (4.12) becomes:

$$\text{Tr}(M_0) = d + 4 + \frac{c_1 \left[2^{d+1} + \left(1 + \frac{1}{c_1} \right)^{d+1} \right] - 2^{d+2}}{1 - c_1} > d + 4,$$

for all $c_1 \in (0, 1)$, and thus the assertion of this theorem follows from Theorem 4.1.

IV. $d = -1$ In the end of this section we analyze the numerical stability of the spline collocation method in the space $S_{m-1}^{(-1)}$, for $m \geq 1$. In this case, when $d = -1$, formula (3.4) becomes:

$$u_n(t_n + \tau h_n) = \sum_{r=1}^m \beta_{n,r} \tau^{r-1}, \text{ for } \tau \in [0, 1], \text{ and } n = 0, 1, \dots, N-1. \quad (4.13)$$

Thus all the coefficients of the approximation $u \in S_{m-1}^{(-1)}$ are determined by the collocation conditions:

$$\begin{aligned} V_0 \beta_0 &= (f(c_0 h_0), \dots, f(c_m h_0))^T, & \text{for } n = 0, \text{ and,} \\ V_n \beta_n &= (1, \dots, 1) u_{n-1}(t_n) + r_n & \text{for } n = 1, 2, \dots, N-1, \end{aligned} \quad (4.14)$$

where the matrix V_n and the vector r_n are like in relation (3.6).

By use of the notations introduced in the case I ($d = 0$), i.e.

$u_{n+1} := (u_n(t_n + c_j h))_{j=1, \overline{m}}^T$, and $E := (c_j^r)_{j,r=1, \overline{m}}$, from equation (4.13) we obtain:

$$u_{n+1} = E \cdot \beta_n, \text{ for } n = 0, \dots, N-1, \quad (4.15)$$

and the elimination of β_n between (4.14) and (4.15) yields:

$$u_{n+1} = (I + O(h)) (1, \dots, 1)^T u_{n-1}(t_n) + (I + O(h)) r_n.$$

The above relation is likewise relation (4.4). Thus by the same reasoning we may prove the following result concerning numerical stability of the $(m, -1)$ -method:

Theorem 4.4. If $d = -1$, then the polynomial spline collocation method is stable for every choice of the collocation parameters $\{c_j\}_{j=1, \overline{m}}$.

5. NUMERICAL EXAMPLE

We give below the results of applying various $(3, d)$ -methods to the following integral linear equation of the second kind:

$$y(t) = (1+t)e^t - \int_0^t e^{t-s} y(s) ds, \text{ for } t \in [0, 1],$$

which has the exact solution $y(t) = e^t$.

In the following we use the notations: $e_1 := |y(t_1) - u(t_1)|$, $e_5 := |y(t_5) - u(t_5)|$, $e_N := |y(1) - u(1)|$, where $u \in S_3^{(d)}$ is the approximated solution and $t_i := ih \in Z_N$. Thus, for $N = 10$ ($h = 0.1$) we obtain:

- a) if the collocation parameters are $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$ and $c_3 = 1$ then we have:
 $e_1 = 0.2 \times 10^{-8}$, $e_5 = 0.8 \times 10^{-8}$, $e_N = 0.2 \times 10^{-7}$ for $d = 0$;
 $e_1 = 0.1 \times 10^{-8}$, $e_5 = 0.1 \times 10^{-8}$, $e_N = 0.15 \times 10^{-7}$ for $d = 1$;
 $e_1 = 0.1 \times 10^{-8}$, $e_5 = 0.2 \times 10^{-6}$, $e_N = 0.2 \times 10^{-3}$ for $d = 2$;

b) if the collocation parameters are the *Radau II - points*, i.e. $c_1 = \frac{4-\sqrt{6}}{10}$, $c_2 = \frac{4+\sqrt{6}}{10}$ and $c_3 = 1$, then we have:

$$e_1 = 1 \times 10^{-10}, e_5 = 0.1 \times 10^{-8}, e_N = 0.3 \times 10^{-8} \text{ for } d = 0;$$

$$e_1 = 0.1 \times 10^{-8}, e_5 = 0.1 \times 10^{-8}, e_N = 0.2 \times 10^{-6} \text{ for } d = 1;$$

$$e_1 = 0.1 \times 10^{-8}, e_5 = 0.2 \times 10^{-4}, e_N = 11.2109 \text{ for } d = 2.$$

From this numerical example we observe that the $(3, d)$ - *method* is stable for $d = 0$ and it is unstable for $d = 2$. The case $d = 1$ will be analyzed in a future paper.

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