



An implicit numerical spline method for systems for ODEs

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Abstract

An implicit numerical method is introduced that gives piecewise polynomial spline approximations for the solution of an initial value problem for the equation

$$y'(t) = f(t, y(t)).$$

Here, f is a function having continuous derivatives up to the order r . The method presented in this work, provides spline functions which approximate the solution of the problem in the most regular function space C^{r+1} . Error estimates in C and C^{r+1} are given and the stability of the method is investigated. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

The spline numerical methods are an accurate instrument for obtaining global approximations of the solutions of ODEs and their derivatives. An analysis of the various spline methods shows that most of them fail to satisfy

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certain types of stability conditions. Consequently, they are not recommended for solving stiff problems, an important and challenging class of ODEs. In the following is presented an implicit spline method and its properties of stability are investigated.

Consider the initial value problem

$$y'(t) = f(t, y(t)) \quad y(0) = \mathbf{y}_0 \quad (1)$$

for $t \in [0, T]$, $f \in C^r([0, T] \times \mathbb{R}^m, \mathbb{R}^m)$, $r \geq 0$. Assume that f satisfies the Lipschitz conditions

$$\|f^{(q)}(t, \mathbf{u}) - f^{(q)}(t, \mathbf{v})\| \leq L \|\mathbf{u} - \mathbf{v}\| \quad (2)$$

for all $t \in [0, T]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and $q = 0, \dots, r$.

In Eq. (2) we define $f^{(q)}$ in the following way:

$$f^{(q)} := f_x^{(q-1)} + f_y^{(q-1)} f, \quad q = 1, \dots, r-1, \quad f^{(0)} := f$$

and by $f_y^{(q-1)}$ we understand the Jacobian of the function $f^{(q-1)}$.

Define on $[0, T]$ the uniform mesh

$$\Delta: 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T,$$

$n \geq 1$ with the step $h := t_{k+1} - t_k$ for all $k = 0, \dots, n-1$. Denote $\mathbf{y}_k^{(j)} := y^{(j)}(t_k)$ for $k = 0, \dots, n$ and $j = 0, \dots, r$.

Assume that on the mesh Δ , $f^{(r)}(\cdot, y(\cdot))$ has the modulus of continuity

$$\omega_r(h) := \max_{i=1, \dots, m} \sup_{|s_1 - s_2| \leq h} |f_i^{(r)}(s_1, y(s_1)) - f_i^{(r)}(s_2, y(s_2))|,$$

which is $O(h)$ because $f \in C^r$.

2. The first approximation process

Let y be the exact solution of the problem. Integrating from t_k to t we get

$$y(t) = \mathbf{y}_k + \int_{t_k}^t f(x, y(x)) \, dx$$

and for $t := t_{k+1}$

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \int_{t_k}^{t_{k+1}} f(x, y(x)) \, dx.$$

The last equality may be approximated by

$$\bar{\mathbf{y}}_{k+1} = \bar{\mathbf{y}}_k + \int_{t_k}^{t_{k+1}} f(x, \mathbf{y}_{k+1}^*(x)) \, dx, \quad (3)$$

where

$$y_{k+1}^*(t) = \sum_{j=0}^{r+1} \frac{\bar{y}_{k+1}^{(j)}}{j!} (t - t_{k+1})^j, \quad t \in [t_k, t_{k+1}], \quad (4)$$

$$\bar{y}_{k+1}^{(j+1)} = f^{(j)}(t_{k+1}, \bar{y}_{k+1}), \quad j = 0, \dots, r. \quad (5)$$

The relation (4) corresponds to the Taylor expansion

$$y(t) = \sum_{j=0}^r \frac{y_{k+1}^{(j)}}{j!} (t - t_{k+1})^j + \frac{\tilde{y}^{(r+1)}(\xi_{k+1})}{(r+2)!} (t - t_{k+1})^{r+2}, \quad t \in [t_k, t_{k+1}].$$

The notation $\tilde{y}^{(r+2)}(\xi_{k+1})$ was used to show that the vectorial function $y^{(r+1)}$ takes different values $t_k < \xi_{k+1} < t_{k+1}$ for each of its components.

Obviously, $\bar{y}_0^{(j)} = y_0^{(j)}$, for $j = 0, \dots, r+1$.

Theorem 1. *The implicit non-linear system in $\bar{y}_{k+1}$ given by Eqs. (3)–(5) is uniquely solvable, provided h sufficiently small.*

Proof. Using Eqs. (3)–(5), the system becomes

$$\bar{y}_{k+1} = \bar{y}_k + \int_{t_k}^{t_{k+1}} f\left(x, \bar{y}_{k+1} + \sum_{j=1}^{r+1} \frac{f^{(j-1)}(t_{k+1}, \bar{y}_{k+1})}{j!} (x - t_{k+1})^j\right) dx$$

Consider the application $\mathcal{A}: \mathbb{R}^m \rightarrow \mathbb{R}^m$,

$$\mathcal{A}(\mathbf{u}) = \bar{y}_k + \int_{t_k}^{t_{k+1}} f\left(x, \mathbf{u} + \sum_{j=1}^{r+1} \frac{f^{(j-1)}(t_{k+1}, \mathbf{u})}{j!} (x - t_{k+1})^j\right) dx$$

Then the system becomes $\mathbf{u} = \mathcal{A}\mathbf{u}$. Consider $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$.

$$\begin{aligned} \|\mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v})\| &\leq \int_{t_k}^{t_{k+1}} \left\| f\left(x, \mathbf{u} + \sum_{j=1}^{r+1} \frac{f^{(j-1)}(t_{k+1}, \mathbf{u})}{j!} (x - t_{k+1})^j\right) \right. \\ &\quad \left. - f\left(x, \mathbf{v} + \sum_{j=1}^{r+1} \frac{f^{(j-1)}(t_{k+1}, \mathbf{v})}{j!} (x - t_{k+1})^j\right) \right\| dx \\ &\leq \int_{t_k}^{t_{k+1}} L \left\| \mathbf{u} + \sum_{j=1}^{r+1} \frac{f^{(j-1)}(t_{k+1}, \mathbf{u})}{j!} (x - t_{k+1})^j - \mathbf{v} \right. \\ &\quad \left. - \sum_{j=1}^{r+1} \frac{f^{(j-1)}(t_{k+1}, \mathbf{v})}{j!} (x - t_{k+1})^j \right\| dx \\ &\leq Lh \|\mathbf{u} - \mathbf{v}\| + L^2 \sum_{j=1}^{r+1} \frac{h^{j+1}}{(j+1)!} \|\mathbf{u} - \mathbf{v}\|. \end{aligned}$$

We observe that the Lipschitz constant of the application $\mathcal{A}$ is

$$L_{\mathcal{A}} = \left(1 + L \sum_{j=1}^{r+1} \frac{h^j}{(j+1)!} \right) Lh \quad (6)$$

and $L_{\mathcal{A}} < 1$ for h sufficiently small. So $\mathcal{A}$ is a contraction and Eqs. (3)–(5) is uniquely solvable, following the Picard–Banach fixed point theorem. $\square$

This result can be significantly improved for linear initial value problems. This is important for stiff problems, where, the very large Lipschitz constants impose severe constraints on the steplength, due to the condition $L_{\mathcal{A}} < 1$ with $L_{\mathcal{A}}$ given by Eq. (6).

Theorem 2. Consider the initial value problem

$$y'(t) = Ay(t) + g(t) \quad y(0) = y_0, \quad (7)$$

with $t \in [0, T]$, $\in \mathbf{M}_{m \times m}(\mathbb{R})$, $g \in C^r([0, T], \mathbb{R}^m)$. Denote $\sigma(A) \in \mathbb{C}$ the set of the m eigenvalues of A . The method (3)–(5) applied to the system (7) has a unique solution for a given step h if and only if

$$\frac{\xi_k}{h} \notin \sigma(A),$$

where ξ_k , $k = 1, \dots, r+2$ are the complex roots of the polynomial

$$P(z) = \sum_{j=0}^{r+2} (-1)^j \frac{z^j}{j!}. \quad (8)$$

Proof. Applying the method (3)–(5) to problem (7) we get the $m(r+2) \times m(r+2)$ linear system

$$\begin{bmatrix} \mathbf{I}_m - hA & \frac{h^2}{2!}A & -\frac{h^3}{3!}A & \cdots & (-1)^{r+2} \frac{h^{r+2}}{(r+2)!}A \\ -A & \mathbf{I}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m \\ -A^2 & \mathbf{0}_m & \mathbf{I}_m & \cdots & \mathbf{0}_m \\ \vdots & \vdots & \vdots & & \vdots \\ -A^{r+1} & \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}}_{k+1} \\ \bar{\mathbf{y}}'_{k+1} \\ \bar{\mathbf{y}}''_{k+1} \\ \vdots \\ \bar{\mathbf{y}}^{(r+1)}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \vdots \\ \mathbf{b}_{r+2} \end{bmatrix}$$

where

$$\mathbf{b}_1 = \bar{\mathbf{y}}_k + \int_{t_k}^{t_{k+1}} g(x) \, dx \quad \mathbf{b}_{i+1} = \sum_{j=0}^i A^{i-j} g^{(j)}(t_{k+1}), \quad i = 1, \dots, r+1.$$

By substitutions we get the $m \times m$ linear system

$$\left(\mathbf{I}_m - hA + \frac{h^2}{2!}A^2 + \cdots + (-1)^{r+2} \frac{h^{r+2}}{(r+2)!}A^{r+2} \right) \bar{\mathbf{y}}_{k+1} = \mathbf{d}, \quad (9)$$

where $\mathbf{d} \in \mathbb{R}^m$ is a free term independent of $\bar{\mathbf{y}}_{k+1}^{(j)}$, $j = 0, \dots, r+1$ or h . Consider $\{\xi_k\}_{k=1, r+2}$ the roots of the polynomial P given by Eq. (8). The matrix

$$M = \mathbf{I}_m - hA + \frac{1}{2!}(hA)^2 + \dots + (-1)^{r+2} \frac{1}{(r+2)!}(hA)^{r+2}$$

of the system (9) can be written as

$$M = \prod_{k=1}^{r+2} (hA - \xi_k \mathbf{I}_m)$$

so the existence and uniqueness condition for the solution of (9) is

$$\det \left[\prod_{k=1}^{r+2} (hA - \xi_k \mathbf{I}_m) \right] \neq 0$$

or, equivalently

$$\prod_{k=1}^{r+2} \det \left(A - \frac{\xi_k}{h} \mathbf{I}_m \right) \neq 0$$

which proves the conclusion of the theorem. $\square$

In the following we obtain error estimates for the first approximation process.

Lemma 1. *The inequality*

$$\|\mathbf{y}_{k+1} - \bar{\mathbf{y}}_{k+1}\| \leq \frac{1}{1 - c_1 h} \|\mathbf{y}_k - \bar{\mathbf{y}}_k\| + \frac{c_2}{1 - c_1 h} \omega_r(h) h^{r+2}$$

holds for $k = 0, \dots, n-1$, where c_1 and c_2 are positive constants, independent of h .

Proof. We estimate

$$\begin{aligned} \|\mathbf{y}_{k+1} - \bar{\mathbf{y}}_{k+1}\| &\leq \|\mathbf{y}_k - \bar{\mathbf{y}}_k\| + L \int_{t_k}^{t_{k+1}} \|y(x) - y_{k+1}^*(x)\| dx \leq \|\mathbf{y}_k - \bar{\mathbf{y}}_k\| \\ &\quad + L \int_{t_k}^{t_{k+1}} \left\| \sum_{j=0}^r \frac{\mathbf{y}_{k+1}^{(j)}}{j!} (x - t_{k+1})^j + \frac{\bar{\mathbf{y}}_{k+1}^{(r+1)}(\xi_{k+1})}{(r+1)!} - \sum_{j=0}^{r+1} \frac{\bar{\mathbf{y}}_{k+1}^{(j)}}{j!} (x - t_{k+1})^j \right\| dx \\ &\leq \|\mathbf{y}_k - \bar{\mathbf{y}}_k\| + L \sum_{j=0}^{r+1} \frac{\|\mathbf{y}_{k+1}^{(j)} - \bar{\mathbf{y}}_{k+1}^{(j)}\|}{(j+1)!} h^{j+1} + L \frac{\omega_r(h)}{(r+2)!} h^{r+2} \\ &\leq \|\mathbf{y}_k - \bar{\mathbf{y}}_k\| + Lh \|\mathbf{y}_{k+1} - \bar{\mathbf{y}}_{k+1}\| \\ &\quad + L \sum_{j=0}^r \frac{\|f^{(j)}(t_{k+1}, \mathbf{y}_{k+1}) - f^{(j)}(t_{k+1}, \bar{\mathbf{y}}_{k+1})\|}{(j+2)!} h^{j+2} + L \frac{\omega_r(h)}{(r+2)!} h^{r+2} \end{aligned}$$

$$\begin{aligned}
&\leq \|y_k - \bar{y}_k\| + Lh\|y_{k+1} - \bar{y}_{k+1}\| + L^2h \sum_{j=0}^r \frac{h^{j+1}}{(j+2)!} \|y_{k+1} - \bar{y}_{k+1}\| \\
&\quad + L \frac{\omega_r(h)}{(r+2)!} h^{r+2} \\
&\leq \|y_k - \bar{y}_k\| + c_1h\|y_{k+1} - \bar{y}_{k+1}\| + L \frac{\omega_r(h)}{(r+2)!} h^{r+2}.
\end{aligned}$$

So, for h small enough,

$$\|y_{k+1} - \bar{y}_{k+1}\| \leq \frac{1}{1 - c_1h} \|y_k - \bar{y}_k\| + \frac{c_2}{1 - c_1h} \omega_r(h) h^{r+2}. \quad \square$$

Theorem 3. *The inequality*

$$\|y_{k+1}^{(j)} - \bar{y}_{k+1}^{(j)}\| \leq c_4 h^{r+1} \omega_r(h)$$

holds for all $k = 0, \dots, n-1$, $j = 0, \dots, r+1$, where c_3 is a positive constant, independent of h .

Proof. From Lemma 2.1, for $j = 0$, multiplying by a suitable factor, we get for all $i \in 0, \dots, k$, $k = 1, \dots, n-1$, that

$$\frac{1}{(1 - c_1h)^{k-i}} \|y_{i+1} - \bar{y}_{i+1}\| \leq \frac{1}{(1 - c_1h)^{k-i+1}} \|y_i - \bar{y}_i\| + \frac{c_2 \omega_r(h)}{(1 - c_1h)^{k-i+1}} h^{r+2}$$

and by summing the previous inequalities from $i = 1$ to $i = k$ we get

$$\begin{aligned}
\|y_{k+1} - \bar{y}_{k+1}\| &\leq c_2 \sum_{i=1}^{k+1} \frac{1}{(1 - c_1h)^i} \omega_r(h) h^{r+2} \\
&\leq c_2 \frac{1}{1 - c_1h} \frac{\frac{1}{(1 - c_1h)^{k+1}} - 1}{\frac{1}{1 - c_1h} - 1} \omega_r(h) h^{r+2} = c_1 c_2 \left(\frac{1}{(1 - c_1h)^{k+1}} - 1 \right) \omega_r(h) h^{r+1}.
\end{aligned}$$

We estimate

$$\frac{1}{(1 - c_1h)^{k+1}} = \left(1 + \frac{c_1h}{1 - c_1h} \right)^{k+1} \leq \left[\left(1 + \frac{c_1T}{n - c_1T} \right)^{n - c_1T} \right]^{\frac{n}{n - c_1T}} \leq e^{c_1T}$$

and taking $c_3 := c_1 c_2 (e^{c_1T} - 1)$ we have

$$\|y_{k+1} - \bar{y}_{k+1}\| \leq c_3 h^{r+1} \omega_r(h).$$

Using Eq. (5) and the Lipschitz conditions (2), we obtain

$$\|y_{k+1}^{(j)} - \bar{y}_{k+1}^{(j)}\| \leq c_4 h^{r+1} \omega_r(h). \quad \square$$

3. The second approximation process

From the first approximation process, the following sets of approximate values have been obtained

$$\bar{\mathbf{Y}}^{(j)} = \{\bar{\mathbf{y}}_0^{(j)}, \dots, \bar{\mathbf{y}}_n^{(j)}\} \in \mathbb{R}^m, \quad j = 0, \dots, r+1.$$

We construct a vector valued spline function s of degree $2r+3$ and class of continuity C^{r+1} that interpolates the values $\bar{\mathbf{Y}}^{(j)}$ on the given mesh Δ .

Theorem 4. *For the given mesh of points Δ and for the given sets of values $\bar{\mathbf{Y}}^{(j)}$, $j = 0, \dots, r+1$ there is a unique vector valued deficient spline function s that interpolates the values $\bar{\mathbf{Y}}^{(j)}$ on the mesh Δ and satisfies the conditions*

- (i) $s \in C^{r+1}([0, T], \mathbb{R}^m)$
- (ii) $s^{(j)}(t_k) = \bar{\mathbf{y}}_k^{(j)}, \quad j = 0, \dots, r+1, \quad k = 0, \dots, n.$
- (iii) For $t_k \leq t \leq t_{k+1}$

$$s(t) = \sum_{j=0}^{r+1} \frac{\bar{\mathbf{y}}_k^{(j)}}{j!} (t - t_k)^j + \sum_{j=1}^{r+2} \mathbf{a}_k^{(j)} (t - t_k)^{r+1+j}, \quad k = 0, \dots, n-1.$$

Proof. To determine the spline function s under the conditions (i), (ii) and (iii) is equivalent with solving the following interpolation problems on $[t_k, t_{k+1}]$, for each $k = 0, \dots, n-1$:

$$s_k \in \Pi_{2r+3}([t_k, t_{k+1}]) \quad s_k^{(j)}(t_i) = \bar{\mathbf{y}}_i^{(j)}, \quad i = k, k+1, \quad j = 0, \dots, r+1. \quad (10)$$

Eq. (10) are full Hermite interpolation problems and the spline function s is obtained by

$$s|_{[t_k, t_{k+1}]} = s_k, \quad k = 0, \dots, n-1.$$

Now we construct the full Hermite interpolator s_k . Let

$$w(t) = (t - t_k)^{r+2} (t - t_{k+1})^{r+2}$$

$$u_i(t) = \frac{w(t)}{(t - t_i)^{r+2}}, \quad i = k, k+1.$$

Consider the full Hermite fundamental polynomials g_{ij} defined by

$$g_{ij}^{(l)}(t_m) = \delta_{i,m-k} \delta_{jl}, \quad i = 0, 1, \quad m = k, k+1 \quad \text{and} \quad j, l = 0, \dots, r+1. \quad (11)$$

The $2r+3$ degree interpolator becomes

$$s_k(t) = \sum_{i=k}^{k+1} \sum_{j=0}^{r+1} g_{ij}(t) \bar{\mathbf{y}}_i^{(j)} =: \sum_{i=k}^{k+1} \mathbf{g}_i(t). \quad (12)$$

Now we can consider the following Taylor expansions for the polynomials $g_i(\cdot)/u_i(\cdot)$:

$$\frac{g_i(t)}{u_i(t)} = \sum_{j=0}^{r+1} \frac{(t-t_i)^j}{j!} \frac{d^j}{dt^j} \left(\frac{g_i(t)}{u_i(t)} \right) \Big|_{t=t_i}. \quad (13)$$

Substituting Eq. (13) in Eq. (12) and taking into account Eq. (11) and Eq. (12) becomes

$$s_k(t) = \sum_{i=k}^{k+1} u_i(t) \sum_{j=0}^{r+1} \frac{(t-t_i)^j}{j!} \sum_{l=0}^j \binom{j}{l} \bar{y}_i^{(l)} \left(\frac{1}{u_i(t)} \right)^{(j-l)} \quad (14)$$

and after rearranging the summation order and indices in Eq. (14) we obtain

$$s_k(t) = \sum_{i=k}^{k+1} \sum_{j=0}^{r+1} \left(u_i(t) \sum_{l=j}^{r+1} \binom{j}{l} \frac{(t-t_i)^l}{l!} \frac{d^{l-j}}{dt^{l-j}} \left(\frac{1}{u_i(t)} \right) \Big|_{t=t_i} \right) \bar{y}_i^{(j)}. \quad (15)$$

Comparing Eq. (12) and Eq. (15) we get an expression for the full Hermite fundamental polynomials

$$g_{ij}(t) = u_i(t) \sum_{l=j}^{r+1} \binom{j}{l} \frac{(t-t_i)^l}{l!} \frac{d^{l-j}}{dt^{l-j}} \left(\frac{1}{u_i(t)} \right) \Big|_{t=t_i}.$$

So the spline function s under the conditions (i), (ii) and (iii) always exists, is uniquely determined and can be explicitly constructed using formula (15). $\square$

Theorem 5. *Let y be the exact solution of Eq. (1) and s the spline constructed in the previous theorem. Then there is a positive constant c_5 independent of h for which the inequalities*

$$\|y^{(j)}(t) - s^{(j)}(t)\| \leq c_5 \omega_r(h) h^{r+1-j}, \quad j = 0, \dots, r+1$$

hold for any $t \in [0, T]$.

Proof. Using Theorem 3 and the approximation properties of the full Hermite interpolation, the conclusion is immediate. $\square$

Corollary 1. *In the conditions of the previous theorems,*

$$\|y - s\|_{C^{r+1}([0, T], \mathbb{R}^m)} = O(h).$$

Remark 1. We constructed a method that uniformly approximates the solution $y \in C^{r+1}([0, T], \mathbb{R}^m)$ of the system (1) and its derivatives with a vector valued deficient spline function $s \in C^{r+1}([0, T], \mathbb{R}^m)$ having a minimal degree $2r+3$. As a consequence of the error estimates, the method has proven a high convergence order, $O(h^{r+1})$, with one order higher than the other similar

methods. Moreover, the last corollary shows that the spline solutions s converge to the exact solution y in the most regular space $C^{r+1}([0, T], \mathbb{R}^m)$.

As the main purpose for the implicit methods is solving stiff systems, we have to investigate different types of stability conditions pertinent to stiffness.

4. Stability

The spline method we introduced is an implicit one and obviously, it is much more costly from the computational point of view than the explicite ones. So we have to apply this spline method to problems which cannot be solved using explicit methods such as stiff problems. Consequently, we have to investigate for our method different types of stability conditions pertinent to stiffness.

The region of stability corresponding to the method is

$$\mathcal{R} = \{z \in \mathbb{C} : |R(z)| < 1\},$$

where $R : \mathbb{C} \rightarrow \mathbb{C}$ is the stability function of the method.

For finding the stability regions, we apply the first approximation process to the following 1D test problem

$$y' = \lambda y, \quad \lambda \in \mathbb{C}, \quad y(0) = y_0.$$

After the calculations we get

$$\bar{y}_{k+1} = \frac{1}{\sum_{j=0}^{r+2} (-1)^j \frac{(\lambda h)^j}{j!}} \bar{y}_k$$

and so, the stability function is

$$R(z) = \frac{1}{\sum_{j=0}^{r+1} (-1)^j \frac{z^j}{j!}}.$$

Definition 1 (Dalquist, 1963). A method is said to be *A-stable* if $\mathcal{R} \supseteq \{z | \operatorname{Re} z < 0\}$.

Definition 2 (Cryer, 1972). A method is said *A_0 -stable* if $\mathcal{R} \supseteq \{z | \operatorname{Re} z < 0, \operatorname{Im} z = 0\}$.

Definition 3 (Gear, 1969). A method is said to be *stiffly stable* if $\mathcal{R} \supseteq \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_1 = \{z | \operatorname{Re} z < -a\}$, $\mathcal{R}_2 = \{z | -a \leq \operatorname{Re} z < 0, -c < \operatorname{Im} z \leq c\}$ and a and c are positive constant real numbers.

Lemma 2. The constructed method is *A_0 -stable* for all $r \geq 0$.

Proof. For $\operatorname{Re} z < 0$, $\operatorname{Im} z = 0$, $(-1)^j z^j > 0$ and so $|R(z)| < 1$. $\square$

Lemma 3. *The constructed method is A -stable if and only if $r = 0$.*

Proof. We observe that in our case $R(z) = \hat{R}_{r+2}^0(z)$: the $(0, r+2)$ Padé approximation of the exponential. The Ehle conjecture, proven by Hairer and Wanner in Ref. [3] gives a necessary and sufficient condition for the A -acceptability of a Padé approximation $\hat{R}_T^S$: $S - 2 \leq T \leq S$. So, the only A -stable method of the constructed type is for $r = 0$ [1–7]. $\square$

Trying to use them for stiff problems and consequently to investigate A - or stiff stability for them leads to solving the following inequality for $z \in \mathbb{C}$:

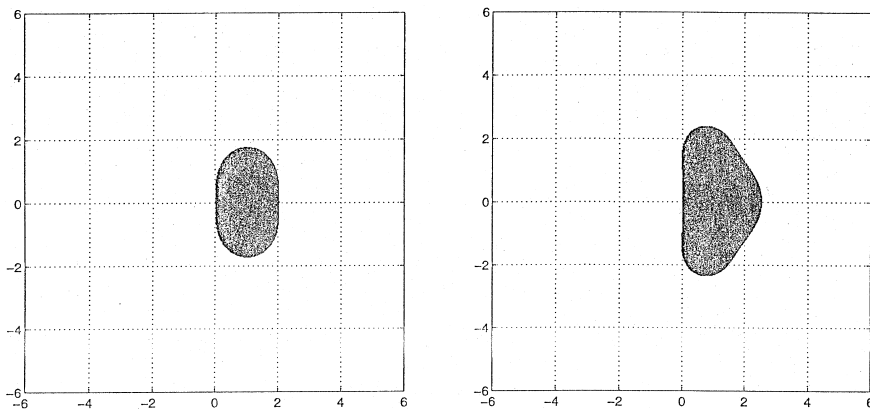


Fig. 1. The regions of stability (in white) for $r=0$ (left) and $r=1$ (right).

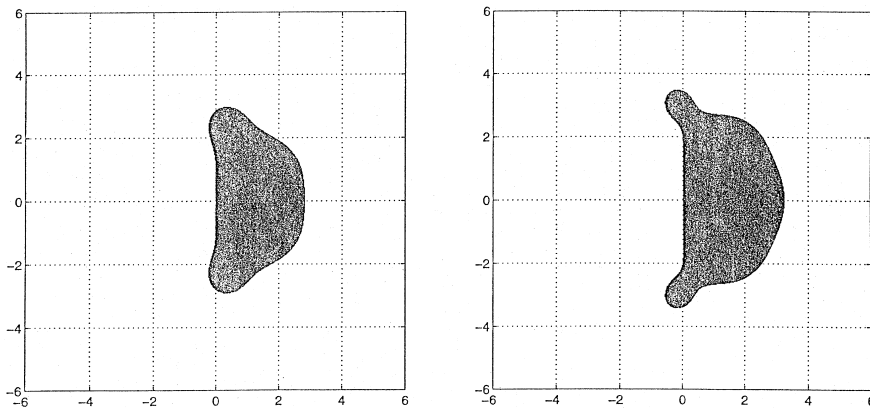


Fig. 2. The regions of stability (in white) for $r=2$ (left) and $r=3$ (right).

$$\left| \sum_{j=0}^{r+2} (-1)^j \frac{z^j}{j!} \right| \geq 1. \quad (16)$$

For $r = 0$ to $r = 7$ we solved numerically Eq. (16) and plotted the solutions in the complex plane.

Figs. 1–4 present the stability regions (in white) for the corresponding spline methods. As one can see, the methods have very good stability properties, stiff stable, with relatively limited instability areas in the left semiplane.

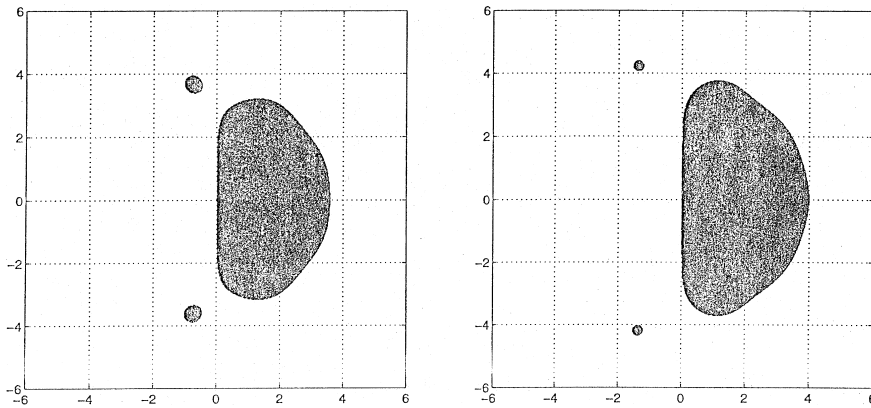


Fig. 3. The regions of stability (in white) for $r = 4$ (left) and $r = 5$ (right).

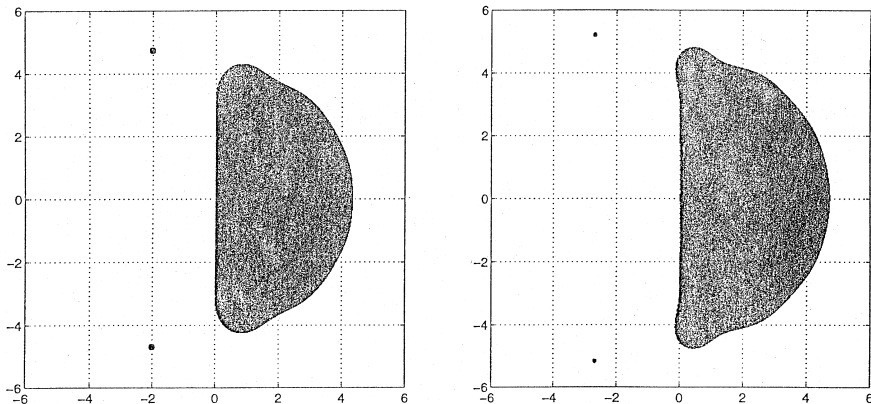


Fig. 4. The regions of stability (in white) for $r = 6$ (left) and $r = 7$ (right).

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