

Extension and approximation of semi-Lipschitz functions on a quasi-metric space

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Abstract

We show that every semi-Lipschitz real function defined on a subset Y of a quasi-metric space (X, d) has at least an extension preserving the smallest semi-Lipschitz constant. If X is a linear space and the quasi-metric d is convex (starshaped), then one obtains extensions preserving both the smallest semi-Lipschitz constant and convexity (starshapedness) of a convex (starshaped) semi-Lipschitz function on Y .

The connections with a special problem of best approximation in the semilinear quasi-normed space $SLip_0 X$ are discussed.

1 Introduction

In the last years there has been a rising interest in the study of quasi-metric and quasi-uniform spaces (see [6], [9]), a fact which allowed the extension of some classic results to this context. A comment about results on this line as well as on applications, can be found in [18], where the authors study the quasi-normed semilinear space $SLip_0 X$ of real-valued semi-Lipschitz functions defined on a quasi-metric space (X, d) .

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They use the elements of $SLip_0 X$ to characterize the elements of best approximation and the semi-Chebyshev subsets of the quasi-metric space (X, d) , extending the results from [11], [13], [16].

Continuing this study of semi-Lipschitz functions, we present in this paper a theorem of extension with the preservation of the quasi-norm of semi-Lipschitz functions defined on a subset Y of X , an analogue of McShane's result (see [1], [5], [10]) on the extension of Lipschitz functions.

We emphasize the connection between the extension of $f|_Y$, for $f \in SLip_0 X$, and the elements of best approximation of f by elements in the annihilator space $Y^\perp$ in $SLip_0 X$ of the set Y , similar to those proved in [4], [12] and [17] in the case of metric spaces and Lipschitz functions. We study also the possibility of the extension of convex (or starshaped) semi-Lipschitz functions, preserving both convexity (or starshapedness) and Lipschitz quasi-norm. In the case of metric spaces results of this kind were proved in [3] and [12].

Let X be a nonvoid set. A *quasi-metric* on X is a function $d : X \times X \rightarrow [0, +\infty)$ satisfying the conditions:

- (i) $d(x, y) = d(y, x) = 0 \iff x = y; x, y \in X$,
- (ii) $d(x, y) \leq d(x, z) + d(z, y), x, y, z \in X$

or the conditions

- (j) $d(x, y) = 0 \iff x = y, x, y \in X$;
- (jj) $d(x, y) \leq d(x, z) + d(z, y), x, y, z \in X$.

The essential difference with respect to a metric on X is that a quasi-metric does not satisfy the symmetry condition $d(x, y) = d(y, x)$ for all $x, y \in X$.

If d takes the value $+\infty$ it is called an *extended quasi-metric* on X .

Each extended quasi-metric d on X induces a topology τ on X which has base the family of balls $\{B(x, r) : x \in X, r > 0\}$ where $B(x, r) = \{y \in X : d(x, y) < r\}$.

If the quasi-metric d verifies (i) and (ii) then the topology τ is T_0 -separated, and if d verifies (j) and (jj) then the topology τ is T_1 -separated. (see [18], [9]).

By a *quasi-metric space* we mean a pair (X, d) where X is a set and d is a quasi-metric on X .

If X is a linear space and d a quasi-metric on X then the pair (X, d) is called a *linear quasi-metric space*.

The *conjugate* of a quasi-metric d , denoted by d^{-1} is defined by $d^{-1}(x, y) = d(y, x)$, $x, y \in X$.

Obviously the function $d^s : X \times X \rightarrow [0, \infty)$ defined by

$$d^s(x, y) = \max \{d(x, y), d^{-1}(x, y)\}; x, y \in X$$

is a metric on X .

Let (X, d) be a quasi-metric space. A function $f : X \rightarrow \mathbb{R}$ is called *semi-Lipschitz* provided there exists a constant $K(f) \geq 0$ such that

$$f(x) - f(y) \leq K(f) d(x, y), \quad (1.1)$$

for all $x, y \in X$.

The number $K(f) \geq 0$ in (1.1) is called a *semi-Lipschitz constant* for f .

For a quasi-metric space (X, d) , the real valued function $f : X \rightarrow \mathbb{R}$ is said to be $\leq_d$ -increasing if

$$d(x, y) = 0 \text{ implies } f(x) - f(y) \leq 0, \quad x, y \in X \quad (1.2)$$

or equivalently,

$$f(x) - f(y) > 0 \text{ implies } d(x, y) > 0, \quad x, y \in X. \quad (1.3)$$

By (1.1) it follows that every semi-Lipschitz function on (X, d) is $\leq_d$ -increasing and every real-valued function defined on a T_1 -separated quasi-metric space (X, d) is $\leq_d$ -increasing.

The definition of a $\leq_d$ -increasing function $f : X \rightarrow \mathbb{R}$ is consistent for T_0 -separated quasi-metric space (X, d) (see [18]).

For a semi-Lipschitz function $f : X \rightarrow \mathbb{R}$, where (X, d) is a quasi-metric space, denote by $\|f\|_X$ the constant

$$\|f\|_X = \sup \left\{ \frac{(f(x) - f(y)) \vee 0}{d(x, y)} : d(x, y) > 0, x, y \in X \right\} \quad (1.4)$$

Theorem 1.1. Let (X, d) a quasi-metric space and $f : X \rightarrow \mathbb{R}$ a semi-Lipschitz function. Then $\|f\|_X$ defined by (1.4) is the smallest semi-Lipschitz constant for f .

Proof. If $f : X \rightarrow \mathbb{R}$ is semi-Lipschitz, then f is $\leq_d$ -increasing, and then

$$f(x) - f(y) > 0 \text{ implies } d(x, y) > 0.$$

It follows that

$$\frac{(f(x) - f(y)) \vee 0}{d(x, y)} = \frac{f(x) - f(y)}{d(x, y)} > 0.$$

The inequalities $f(x) - f(y) \leq 0$ and $d(x, y) > 0$ imply

$$\frac{(f(x) - f(y)) \vee 0}{d(x, y)} = 0.$$

Consequently $\|f\|_X \geq 0$.

For $f(x) - f(y) < 0$ it follows

$$\frac{f(x) - f(y)}{d(x, y)} \leq \|f\|_X$$

and, evidently, for $f(x) - f(y) \leq 0$ we have $f(x) - f(y) \leq 0 \leq \|f\|_X d(x, y)$.

Consequently we have

$$f(x) - f(y) \leq \|f\|_X d(x, y)$$

for all $x, y \in X$.

Now let $K = K(f) \geq 0$ be such that

$$f(x) - f(y) \leq K d(x, y), \text{ for all } x, y \in X.$$

Since the function f is $\leq_d$ -increasing then

$$\frac{(f(x) - f(y)) \vee 0}{d(x, y)} = \begin{cases} \frac{f(x) - f(y)}{d(x, y)} \leq K & \text{if } f(x) - f(y) > 0 \\ 0 \leq K & \text{if } f(x) - f(y) \leq 0 \end{cases}$$

Taking the supremum with respect to $d(x, y) \neq 0$ it follows $\|f\|_X \leq K(f)$. $\square$

The set of all real semi-Lipschitz function defined on the quasi-metric space (X, d) is denoted by:

$$SLip X = \{f : X \rightarrow \mathbb{R} \mid f \text{ is } \leq_d \text{-increasing, } \|f\|_X < \infty\} \quad (1.5)$$

2 Extensions of semi-Lipschitz functions

Let (X, d) be a quasi-metric space and $Y \subset X$, $Y \neq \emptyset$. Then (Y, d) is a quasi-metric space with the quasi-metric induced by d (denoted by d too).

If $f : Y \rightarrow \mathbb{R}$ is semi-Lipschitz on Y denote by $\|f\|_Y$ the smallest semi-Lipschitz constant for f on Y (see (4)) and

$$SLip Y = \{f : Y \rightarrow \mathbb{R} \mid f \text{ is } \leq_d \text{-increasing, } \|f\|_Y < \infty\}. \quad (2.1)$$

If $f \in SLip Y$, a function $F \in SLip X$ is called an *extension* (preserving the smallest semi-Lipschitz constant) if

$$F|_Y = f \text{ and } \|F\|_X = \|f\|_Y. \quad (2.2)$$

Denote by $E_Y(f)$ the set of all extensions of the function $f \in SLip Y$, i.e.,

$$E_Y(f) = \{F \in SLip X \mid F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\}. \quad (2.3)$$

Theorem 2.1. Let (X, d) a quasi-metric space and Y a nonvoid subset of X . Then for every $f \in SLip Y$ the set $E_Y(f)$ is nonvoid.

Proof. Let $f \in SLip Y$ and the constant $\|f\|_Y$ be defined by (4).

Consider the function

$$F(x) = \inf_{y \in Y} \{f(y) + \|f\|_Y d(x, y)\}, \quad x \in X. \quad (2.4)$$

a) We show first that F is well defined.

Let $z \in Y$ and $x \in X$. For any $y \in Y$ we have

$$f(y) + \|f\|_Y d(x, y) = f(z) + \|f\|_Y d(x, y) - (f(z) - f(y)) \geq$$

$$\begin{aligned} &\geq f(z) + \|f\|_Y d(x, y) - \|f\|_Y d(z, y) = \\ &= f(z) - \|f\|_Y (d(z, y) - d(x, y)). \end{aligned}$$

The inequality $d(z, y) - d(x, y) \leq d(z, x) = d^{-1}(x, z)$ implies

$$f(y) + \|f\|_Y d(x, y) \geq f(z) - \|f\|_Y \cdot d^{-1}(x, z) \quad (2.5)$$

showing that for every $x \in X$ the set $\{f(y) + \|f\|_Y d(x, y) : y \in Y\}$ is bounded from above by $f(z) - \|f\|_Y d^{-1}(x, z)$, and the infimum (9) is finite.

b) We show now that $F(y) = f(y)$ for all $y \in Y$.

Let $y \in Y$. Then

$$F(y) \leq f(y) + \|f\|_Y d(y, y) = f(y).$$

For any $v \in Y$ we have

$$f(y) - f(v) \leq \|f\|_Y \cdot d(y, v)$$

so that

$$f(v) + \|f\|_Y \cdot d(y, v) \geq f(y)$$

and

$$F(y) = \inf \{f(v) + \|f\|_Y d(y, v) : v \in Y\} \geq f(y).$$

It follows $F(y) = f(y)$.

c) We prove that $\|F\|_X = \|f\|_Y$.

Since $F|_Y = f$, the definitions of $\|F\|_X$ and $\|f\|_Y$ yield $\|F\|_X \geq \|f\|_Y$.

Let $x_1, x_2 \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

$$F(x_1) \geq f(y) + \|f\|_Y d(x_1, y) - \varepsilon$$

we obtain

$$\begin{aligned} F(x_2) - F(x_1) &\leq f(y) + \|f\|_Y d(x_2, y) - (f(y) + \|f\|_Y d(x_1, y) - \varepsilon) \\ &= \|f\|_Y [d(x_2, y) - d(x_1, y)] + \varepsilon \end{aligned}$$

$$\leq \|f\|_Y \cdot d(x_2, x_1) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary it follows

$$F(x_2) - F(x_1) \leq \|f\|_Y \cdot d(x_2, x_1)$$

for any $x_1, x_2 \in X$ and $\|F\|_X \leq \|f\|_Y$.

d) The function F is $\leq_d$ -increasing.

Indeed, let be $u, v \in X$ and $d(u, v) = 0$. We have $d(u, y) \leq d(u, v) + d(v, y)$. Consequently

$$d(u, y) \leq d(v, y).$$

Then

$$f(y) + \|f\|_Y d(u, y) \leq f(y) + \|f\|_Y d(v, y).$$

It follows that

$$F(u) \leq F(v),$$

and consequently $d(u, v) = 0$ implies $F(u) \leq F(v)$.

It follows that $F \in E_Y(f)$ so that $E_Y(f) \neq \emptyset$. □

Remarks. ¹⁰ Similarly, the function

$$G(x) = \sup_{y \in Y} \{f(y) - \|f\|_Y d^{-1}(x, y)\} \quad (2.6)$$

is $\leq_d$ -increasing, and G belongs to $E_Y(f)$ too.

²⁰ The inequality

$$G(x) \leq F(x), \quad (2.7)$$

holds for every $x \in X$.

Indeed, taking the infimum with respect to $z \in Y$ and then the supremum with respect to $y \in Y$ in (10) we find

$$G(x) = \sup_{y \in Y} \{f(y) - \|f\|_Y d^{-1}(x, y)\} \leq \inf_{z \in Y} \{f(z) + \|f\|_Y d(x, z)\} = F(x).$$

In fact, it holds the following theorem:

Theorem 2.2. Let (X, d) be a quasi-metric space, Y a nonvoid subset of X and $f \in SLip Y$.

Then for any $H \in E_Y(f)$ we have

$$G(x) \leq H(x) \leq F(x), \quad x \in X. \quad (2.8)$$

Proof. Let $H \in E_Y(f)$. For arbitrary $x \in X$ and $y \in Y$ we have

$$H(x) - H(y) \leq \|f\|_Y d(x, y)$$

implying

$$H(x) \leq H(y) + \|f\|_Y d(x, y) = f(y) + \|f\|_Y d(x, y).$$

Taking the infimum with respect to $y \in Y$ we get

$$H(x) \leq \inf_{y \in Y} \{f(y) + \|f\|_Y d(x, y)\} = F(x).$$

The inequality $H(x) \geq G(x)$, $x \in X$ can be proved similarly. □

Example 2.3. ¹⁰ Let $\mathbb{R}$ be the real axis and $d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ the quasi-metric defined by

$$d(x, y) = \begin{cases} x - y, & \text{if } x \geq y \\ 1, & \text{if } x < y \end{cases}$$

Then the quasi-metric space $(\mathbb{R}, d)$ is T_1 -separated and it is called the Sorgenfrey line. (see [18]).

Let Y be given by $Y = [0, 1]$ and $f : Y \rightarrow \mathbb{R}$ $f(y) = 2y$. Then f is semi-Lipschitz on Y and $\|f\|_Y = 2$. The extension F defined by (2.4) is

$$F(x) = \begin{cases} 2, & \text{if } x < 0, \\ 2x, & \text{if } x \geq 0, \end{cases}$$

and the extension G defined by (2.6) is

$$G(x) = \begin{cases} 2x & \text{if } x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Observe that $F(x) \geq G(x)$ for all $x \in \mathbb{R}$ and F, G are lower semicontinuous.

In fact, every real semi-Lipschitz function is lower semicontinuous.

²⁰ An interesting quasi-metric space is the Khalimsky line, i.e. the set $\mathbb{Z}$ of integers endowed with the quasi-metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 0 & \text{if } x = 2n \text{ and } y = 2n + 1 \text{ or } y = 2n - 1, n \in \mathbb{Z}, \\ 1, & \text{otherwise} \end{cases}$$

(see [18]).

3 Extension of convex semi - Lipschitz functions

Let X be a linear space and d a quasi-metric on X . If Y is a nonvoid subset of X , by Theorem 2, for every $f \in SLip Y$ the set of extensions $E_Y(f)$ preserving the smallest semi-Lipschitz constant $\|f\|_Y$ is nonvoid.

The following problem arises naturally: which other properties of the function f (beside the semi-Lipschitz constant) are preserved by at least one of its extensions?

In the following it is shown that two such properties are convexity and starshapedness.

Let (X, d) be a linear quasi-metric space and Y a nonvoid convex subset of X (i.e. $(\forall) u_1, u_2 \in Y, (\forall) \alpha \in [0, 1], \alpha u_1 + (1 - \alpha) u_2 \in Y$).

Definition 3.1. If Y is a nonvoid convex of X , where (X, d) is a linear quasi-metric space, the quasi-metric d is called convex on Y if it satisfies the inequality:

$$d(\alpha x_1 + (1 - \alpha) x_2, \alpha y_1 + (1 - \alpha) y_2) \leq \alpha \cdot d(x_1, y_1) + (1 - \alpha) \cdot d(x_2, y_2) \quad (3.1)$$

for all $x_1, x_2, y_1, y_2 \in Y$ and all $\alpha \in [0, 1]$.

If d is convex on X and the set Y is convex, then d is convex on Y .

Theorem 3.2. Let (X, d) be a quasi-metric linear space, d be convex, and let Y be a nonvoid convex subset of X .

- a) If $f \in SLip Y$ is convex on Y then there exists a convex $F \in E_Y(f)$;
- b) If $f \in SLip Y$ is concave (i.e. $-f$ is convex) on Y then there exists a concave $G \in E_Y(f)$.

Proof. Let $f \in SLip Y$ be convex on the convex set Y . Consider the function $F : X \rightarrow \mathbb{R}$ defined by

$$F(x) = \inf_{y \in Y} \{f(y) + \|f\|_Y \cdot d(x, y)\}, \quad x \in X. \quad (3.2)$$

Then F is well defined and $F \in E_Y(f)$ (Theorem 2).

Since Y is convex and the quasi-metric d is convex (in the sense of Definition 1) we have

$$\begin{aligned} F(\alpha x_1 + (1 - \alpha) x_2) &\leq f(\alpha y_1 + (1 - \alpha) y_2) + \|f\|_Y \cdot \\ &\quad \cdot d(\alpha x_1 + (1 - \alpha) x_2, \alpha y_1 + (1 - \alpha) y_2) \\ &\leq \alpha f(y_1) + (1 - \alpha) f(y_2) + \\ &\quad + \|f\|_Y [\alpha d(x_1, y_1) + (1 - \alpha) d(x_2, y_2)] = \\ &= \alpha [f(y_1) + \|f\|_Y d(x_1, y_1)] + \\ &\quad + (1 - \alpha) [f(y_2) + \|f\|_Y d(x_2, y_2)], \end{aligned}$$

for all $x_1, x_2 \in X, y_1, y_2 \in Y$ and $\alpha \in [0, 1]$.

Taking the infimum with respect to $y_1, y_2 \in Y$ we obtain

$$F(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha F(x_1) + (1 - \alpha) F(x_2) \quad (3.3)$$

for all $x_1, x_2 \in X$ and all $\alpha \in [0, 1]$, showing that the function F in $E_Y(f)$, defined by (3.2), is convex.

b) If $f \in SLip Y$ is concave on Y , let $G : X \rightarrow \mathbb{R}$ be defined by

$$G(x) = \sup_{y \in Y} \{f(y) - \|f\|_Y \cdot d(y, x)\}, \quad x \in X \quad (3.4)$$

Then G is well defined and $G \in E_Y(f)$ (Theorem 2).

For any $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\alpha \in [0, 1]$ we have

$$\begin{aligned} G(\alpha x_1 + (1 - \alpha) x_2) &\geq f(\alpha y_1 + (1 - \alpha) y_2) - \\ &\quad - \|f\|_Y d(\alpha y_1 + (1 - \alpha) y_2, \alpha x_1 + (1 - \alpha) x_2) \\ &\geq \alpha f(y_1) + (1 - \alpha) f(y_2) - \\ &\quad - \|f\|_Y [\alpha \cdot d(y_1, x_1) + (1 - \alpha) d(y_2, x_2)] \\ &= \alpha [f(y_1) - \|f\|_Y d(y_1, x_1)] + \\ &\quad + (1 - \alpha) [f(y_2) - \|f\|_Y d(y_2, x_2)] \end{aligned}$$

Taking the supremum with respect to $y_1, y_2 \in Y$ we get

$$G(\alpha x_1 + (1 - \alpha) x_2) \geq \alpha G(x_1) + (1 - \alpha) G(x_2) \quad (3.5)$$

for all $x_1, x_2 \in X$ and $\alpha \in [0, 1]$, showing that the function G from $E_Y(f)$, defined by (3.4), is concave. $\square$

Definition 3.3. Let X be a real linear space and $\theta \in Z \subset X$ where θ denotes the null element of X . The set Z is called starshaped if

$$(\forall) \alpha, \alpha \in [0, 1], (\forall) z \in Z, \alpha z \in Z. \quad (3.6)$$

Obviously that any convex subset of X which contains θ is starshaped, and the converse is not true in general.

A function $f: Z \rightarrow \mathbb{R}$, where Z is a starshaped subset of a linear space X , is called starshaped if

$$f(\alpha z) \leq \alpha f(z). \quad (3.7)$$

for all $z \in Z$ and $\alpha \in [0, 1]$. A convex function $f: Y \rightarrow \mathbb{R}$, defined on a convex subset Y of X containing θ and such that $f(\theta) \leq 0$ is starshaped but there are starshaped functions on Y which are not convex.

Definition 3.4. Let (X, d) be a quasi-metric linear space. The quasi-metric d is called starshaped on X if the inequality

$$d(\alpha x, \alpha y) \leq \alpha d(x, y) \quad (3.8)$$

holds for all $x, y \in X$ and $\alpha \in [0, 1]$.

Remark 3.5. If (X, d) is a quasi-metric linear space with convex quasi-metric, then d is starshaped because the inequality

$$d(\alpha x + (1 - \alpha) \theta), \alpha y + (1 - \alpha) \theta \leq \alpha d(x, y) + (1 - \alpha) d(\theta, \theta)$$

yields

$$d(\alpha x, \alpha y) \leq \alpha d(x, y)$$

for all $x, y \in X$ and $\alpha \in [0, 1]$.

Now we shall prove the extension result for starshaped semi-Lipschitz functions.

Theorem 3.6. Let (X, d) be a quasi-metric linear space with starshaped quasi-metric d and let Z be a starshaped subset of X .

Then every starshaped function $\varphi \in SLipZ$ admits at least one starshaped extension $\Phi \in E_Z(\varphi)$.

Proof. Let $\varphi \in SLipZ$ be starshaped on the starshaped set $Z \subset X$. The function

$$\Phi(x) = \inf_{z \in Z} \{\varphi(z) + \|\varphi\|_Z d(x, z)\}, \quad x \in X, \quad (3.9)$$

belongs to $E_Z(\varphi)$ (Theorem 2).

We shall show that Φ is also starshaped on X . To this end let $x \in X, z \in Z$ and $\alpha \in [0, 1]$. We have

$$\begin{aligned} \Phi(\alpha x) &\leq \varphi(\alpha z) + \|\varphi\|_Z d(\alpha x, \alpha z) \leq \\ &\leq \alpha \varphi(z) + \alpha \|\varphi\|_Z d(x, z) = \\ &= \alpha [\varphi(z) + \|\varphi\|_Z d(x, z)]. \end{aligned}$$

Taking the infimum with respect to $z \in Z$ we get

$$\Phi(\alpha x) \leq \alpha \Phi(x)$$

for all $x \in X$ and all $\alpha \in [0, 1]$, showing that the function Φ defined by (3.9) is a starshaped extension of φ . $\square$

4 Extension of bounded starshaped semi-Lipschitz functions.

Let (X, d) be a quasi-metric linear space with the quasi-metric d starshaped, let Y be a starshaped subset of X and let $f \in SLip_0 Y = \{h \in SLip Y, h(\theta) = 0\}$. We consider the following problem. If the sup-norm of f , denoted by $\|f\|_\infty$, is finite, find $F \in SLip_0 X = \{H \in SLip X, H(\theta) = 0\}$ such that

$$F|_Y = f, \|F\|_X = \|f\|_Y \text{ and } \|F\|_\infty = \|f\|_\infty. \quad (4.1)$$

Observe that if Y is a subspace of X and $f \in SLip_0 Y$ is starshaped then it is possible that f be unbounded on Y .

More exactly we have:

Lemma 4.1. *Let (X, d) be a linear quasi-metric space, Y a subspace of X and $f \in SLip_0 Y$ be starshaped. If there exists $x_0 \in Y$ such that $f(x_0) > 0$ then f is unbounded on Y .*

Proof. For every $x \in Y, x \neq \theta$ the function $h : (0, \infty) \rightarrow \mathbb{R}$ defined by $h(t) = f(tx)/t$ is non-increasing. Indeed, if $0 < t_1 < t_2$ then

$$h(t_1) = \frac{f(t_1 x)}{t_1} = \frac{f(t_2^{-1} \cdot t_1(t_2 x))}{t_1} \leq \frac{t_1}{t_2} \cdot \frac{f(t_2 x)}{t_1} = \frac{f(t_2 x)}{t_2} = h(t_2).$$

In particular, for $t > 1$ and $x_0 \in Y$ with $f(x_0) > 0$ we have $0 < f(x_0) < \frac{f(tx_0)}{t}$ so that, $f(tx_0) \geq t \cdot f(x_0)$ for every $t > 1$, which shows that f is unbounded on Y . $\square$

The following theorem answers positively the question of the extension of bounded starshaped semi-Lipschitz functions, vanishing at θ .

Theorem 4.2. *Let (X, d) be a linear quasi-metric space and Y a starshaped subset of X . Suppose that the quasi-metric d is starshaped on X .*

Let $f \in SLip_0 Y$ be bounded and starshaped. In order to exist $F \in SLip_0 X$ bounded and starshaped such that

$$F|_Y = f, \|F\|_X = \|f\|_Y \text{ and } \|F\|_\infty = \|f\|_\infty$$

it is necessary and sufficient that $f(y) \leq 0$ for all $y \in Y$.

Proof. Sufficiency. Suppose that $f(y) \leq 0$ for all $y \in Y$. Then the function

$$H(x) = \inf_{y \in Y} [f(y) + \|f\|_Y d(x, y)], x \in X$$

is starshaped and $H|_Y = f$ and $\|H\|_X = \|f\|_Y$. (see Theorem 2 and Theorem 10).

Consider the function

$$F(x) = \begin{cases} H(x) & \text{if } H(x) \leq 0 \\ 0 & \text{if } H(x) > 0 \end{cases} \quad (4.2)$$

Since $F|_Y = H|_Y = f$, $\|H\|_X = \|f\|_Y$ and $\|H\|_X \geq \|F\|_X$ it follows that $\|F\|_X = \|f\|_Y$. Therefore $F \in E_Y(f)$.

Let $x \in X$. If $H(x) \leq 0$ then

$$H(\alpha x) \leq \alpha H(x) \leq 0$$

for any $\alpha \in [0, 1]$ so that

$$F(\alpha x) = H(\alpha x) \leq \alpha H(x) = \alpha F(x).$$

If $H(x) > 0$ then $F(x) = 0$. Let $\alpha \in (0, 1)$. If $H(\alpha x) > 0$ then

$$F(\alpha x) = 0 = \alpha F(x).$$

If $H(\alpha x) \leq 0$ then $F(\alpha x) = H(\alpha x)$ and

$$F(\alpha x) = H(\alpha x) \leq 0 = \alpha F(x).$$

It follows that the function F defined by (4.2) is an extension of f which is starshaped and has the same semi-Lipschitz constant as f .

Since

$$\sup \{|F(x)| : x \in X\} \geq \sup \{|F(y)| : y \in Y\} = \|f\|_\infty$$

it follows $\|F\|_\infty \geq \|f\|_\infty$.

Let $x \in X$. For any $y \in Y$ we have

$$f(y) \leq f(y) + \|f\|_Y \cdot d(x, y)$$

which implies

$$\inf_{y \in Y} f(y) \leq \inf_{y \in Y} [f(y) + \|f\|_Y d(x, y)] = H(x).$$

If $H(x) \leq 0$ then

$$\inf_{y \in Y} f(y) \leq H(x) = F(x).$$

Since $f(y) \leq 0$ for all $y \in Y$ we have $\|f\|_\infty = -\inf f(Y)$ so that

$$-\|f\|_\infty \leq F(x) \iff \|f\|_\infty \geq -F(x).$$

If $H(x) > 0$ then

$$\|f\|_\infty \geq 0 = F(x).$$

Using the fact that $F(x) \leq 0$, for all $x \in X$, and the above inequalities we obtain

$$\|f\|_\infty \geq \sup_{x \in X} (-F(x)) = \|F\|_\infty.$$

It follows that $\|F\|_\infty = \|f\|_\infty$.

Necessity. Suppose there exists $y \in Y$ such that $f(y) > 0$. By Lemma 11 f has no bounded starshaped extensions to X . $\square$

5 Uniqueness of the extension of semi-Lipschitz functions on quasi-metric space

Let (X, d) be a quasi-metric space and Y a nonvoid subset of X . By Theorem 2 it follows that for every $f \in SLip Y$ the set $E_Y(f) \subset SLip X$ is nonvoid, and by inequality (2.8) it follows that $\text{card } E_Y(f) = 1$ iff

$$\sup_{y \in Y} [f(y) - \|f\|_Y d(y, x)] = \inf_{y \in Y} [f(y) + \|f\|_Y d(x, y)] \quad (5.1)$$

for all $x \in X$. (see inequality (2.8) and the formulae (2.4) and (2.6)).

Denoting $d(x, Y) = \inf \{d(x, y) : y \in Y\}$ and $d^{-1}(x, Y) = \inf \{d(y, x) : y \in Y\}$, $x \in X$, and using the inequalities

$$\begin{aligned} \sup_{y \in Y} f(y) - \|f\|_Y d^{-1}(x, Y) &\geq \sup_{y \in Y} [f(y) - \|f\|_Y \cdot d(y, x)] \\ \inf_{y \in Y} f(y) + \|f\|_Y d(x, Y) &\leq \inf_{y \in Y} [f(y) + \|f\|_Y d(x, y)], \end{aligned}$$

from (5.1) it follows

$$d(x, Y) + d^{-1}(x, Y) \leq \frac{1}{\|f\|_Y} \left(\sup_{y \in Y} f(y) - \inf_{y \in Y} f(y) \right) \quad (5.2)$$

if $\|f\|_Y > 0$.

If $\|f\|_Y = 0$, by (5.1) it follows that $f(y) \equiv \text{constant on } Y$.

Evidently, for every nonvoid subset Y of X and every constant function c on Y it follows that $\text{card } E_Y(c) = 1$. Indeed

$$\begin{aligned} F(x) &= \inf_{y \in Y} \{c + 0 \cdot d(x, y)\} = c \\ G(x) &= \sup_{y \in Y} \{c - 0 \cdot d(y, x)\} = c \end{aligned}$$

by (2.3) so that $\text{card } E_Y(c) = 1$.

Definition 5.1. The nonvoid subset Y of the quasi-metric space (X, d) is called *bidense* if for every $x \in X$ and every $\varepsilon > 0$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$ and $d(y, x) < \varepsilon$.

Lemma 5.2. If Y is a bidense subset of (X, d) then every $f \in SLip Y$ has a unique extension $F \in SLip X$ (i.e. $\text{card } E_Y(f) = 1$).

Proof. Suppose that there exists $f \in SLip Y$ such that $\text{card } E_Y(f) > 1$. Then by (2.8) it follows that there exists $x_0 \in X$ such that

$$F(x_0) = \inf [f(y) + \|f\|_Y d(x_0, y)] = \sup [f(y) - \|f\|_Y \cdot d(y, x_0)] + \alpha =$$

$$= G(x_0) + \alpha$$

where $\alpha > 0$.

By the bidensity of Y , for every $\varepsilon > 0$ there exists $y' \in Y$ such that $d(x_0, y') < \varepsilon$ and $d(y', x_0) < \varepsilon$. Consequently

$$\begin{aligned} f(y') + \|f\|_Y \cdot \varepsilon &> f(y') + \|f\|_Y d(x_0, y') \geq \inf_{y \in Y} [f(y) + \|f\|_Y d(x_0, y)] \\ &= F(x_0) = G(x_0) + \alpha = \sup_{y \in Y} [f(y) - \|f\|_Y \cdot d(y, x_0)] + \alpha \\ &\geq f(y') - \|f\|_Y d(y', x_0) + \alpha > f(y') - \|f\|_Y \cdot \varepsilon + \alpha. \end{aligned}$$

It follows that

$$f(y') + \|f\|_Y \varepsilon > f(y') - \|f\|_Y \cdot \varepsilon + \alpha$$

and then

$$2 \cdot \|f\|_Y \cdot \varepsilon > \alpha,$$

for every $\varepsilon > 0$. This is a contradiction. $\square$

Theorem 5.3. Let (X, d) be a quasi-metric space, where d satisfies the conditions (j) and (jj) and let $Y \neq X$ be a subset of X containing at least one cluster point. If each function $f \in SLip Y$ has a unique extension then Y is bidense.

Proof. Let $y_0 \in Y$ be a cluster point of the set Y and let $y_n \in Y \setminus \{y_0\}$ $n = 1, 2, \dots$ be such that $\lim_{n \rightarrow \infty} d(y_n, y_0) = 0$.

Claim: There exists $x_0 \in X$ such that $d(x_0, y_0) > 0$ and $d(x_0, y_n) > 0$, $n = 1, 2, 3, \dots$

Indeed, if contrary, then for every $x \in X$, $d(x, y_0) = 0$ or $d(x, y_n) = 0$ for all $n \in \mathbb{N}$.

By condition (j), in the first case $x = y_0 \in Y$ and in the second one $x = y_n \in Y$. It follows $Y = X$, a contradiction.

Consider the function $f : X \rightarrow \mathbb{R}$ defined by

$$f(x) = d(x, y_0) - d(x_0, y_0), \quad x \in X.$$

We have

$$\begin{aligned} f(y_0) &= d(y_0, y_0) - d(x_0, y_0) = -d(x_0, y_0) < 0 \\ f(y_n) &= d(y_n, y_0) - d(x_0, y_0) > -d(x_0, y_0) \end{aligned}$$

for all $n = 1, 2, \dots$. Define the sequence of functions $\varphi_n : f(X) \rightarrow [0, 1]$ by

$$\varphi_n(t) = \begin{cases} 1 & \text{if } t < f(y_0) \\ \frac{f(y_n) - t}{f(y_n) - f(y_0)} & t \in [f(y_0), f(y_n)] \\ 0 & t > f(y_n) \end{cases}$$

The functions $\Psi_n = \varphi_n \circ f : X \rightarrow \mathbb{R}$, $n = 1, 2, \dots$, satisfy

$$\|\Psi_n|_Y\|_Y \geq \frac{(\varphi_n(f(y_0)) - \varphi_n(f(y_n))) \vee 0}{d(y_0, y_n)} = \frac{1}{d(y_0, y_n)} \rightarrow \infty,$$

for $n \rightarrow \infty$.

By the inequality (5.2)

$$d(x, Y) + d^{-1}(x, Y) \leq \frac{1 - 0}{\|\Psi_n|_Y\|_Y} \rightarrow 0$$

for $n \rightarrow \infty$, showing that Y is dense in X , with respect to the quasi-metric d and with respect to d^{-1} , as well. $\square$

6 Extensions and best approximations

Let (X, d) be a quasi-metric space, $x_0 \in X$ a fixed point and let

$$SLip_0 X = \{f \in SLip X : f(x_0) = 0\} \quad (6.1)$$

be the set of all real-valued semi-Lipschitz functions defined on X , vanishing at x_0 .

Let V be a nonvoid set and $\mathbb{R}^+ = [0, \infty)$. Suppose that on V is defined an operation

$$+ : V \times V \rightarrow V$$

such that $(V, +)$ is an Abelian semigroup, i.e. the operation $+$ satisfies the conditions

$$(i) (x + y) + z = x + (y + z)$$

$$(ii) x + y = y + x$$

(iii) $0 + x = x$ (0 is the neutral element of semigroup $(V, +)$)
for all $x, y, z \in V$, and an operation

$$\cdot : \mathbb{R}^+ \times V \rightarrow V$$

having the properties

$$(i) a \cdot (b \cdot x) = (a \cdot b) \cdot x, \quad a, b \in \mathbb{R}^+; x \in V$$

$$(ii) (a + b) \cdot x = (a \cdot x) + (b \cdot x), \quad a, b \in \mathbb{R}^+; x \in V$$

$$(iii) a \cdot (x + y) = a \cdot x + a \cdot y, \quad a \in \mathbb{R}^+; x, y \in V$$

$$(iv) 1 \cdot x = x, \quad 1 \in \mathbb{R}^+; x \in V$$

The system $(V, +, \cdot, \mathbb{R}^+)$ is called a *semilinear space*.

The opposite element (if exists) of $x \in V$ is denoted by $-x$.

A functional $\|\cdot\|_V : V \rightarrow [0, \infty)$ defined on a semilinear space $(V, +, \cdot, \mathbb{R}^+)$ is called a *quasi-norm* on V if it satisfies the conditions:

$$(i) x, -x \in V \text{ and } \|x\|_V = \|-x\|_V = 0 \iff x = 0$$

$$(ii) \|ax\|_V = a \|x\|_V, \quad a \in \mathbb{R}^+, x \in V$$

$$(iii) \|x + y\|_V \leq \|x\|_V + \|y\|_V, \quad x, y \in V.$$

The pair $(V, \|\cdot\|_V)$ is called a *quasi-normed semilinear space* (see [18] and [9]).

It is immediate that the functional defined by (1.4) is a quasi-norm on $SLip_0 X$, i.e., the pair $(SLip_0 X, \|\cdot\|_X)$ is a quasi-normed semilinear space ([18]).

If $Y \subset X$ and $x_0 \in Y$ then one considers the semi-Lipschitz functions on Y which vanish at x_0 and the quasi-normed semilinear space $(SLip_0 Y, \|\cdot\|_Y)$, where $\|\cdot\|_Y$ is defined like in (1.4) with Y instead of X .

By Theorem 2, for every $f \in SLip_0 Y$, the set of extensions $E_Y(f) \subset SLip_0 X$ is nonvoid. This set is also convex, because if $F_1, F_2 \in E_Y(f)$ and $\alpha \in (0, 1)$ then

$$(\alpha F_1 + (1 - \alpha) F_2)|_Y = \alpha f + (1 - \alpha) f = f$$

and

$$\begin{aligned} \|\alpha F_1 + (1 - \alpha) F_2\|_X &\leq \alpha \|F_1\|_X + (1 - \alpha) \|F_2\|_X = \\ &= \alpha \|f\|_Y + (1 - \alpha) \|f\|_Y = \|f\|_Y. \end{aligned}$$

On the other hand

$$\begin{aligned} \|f\|_Y &= \|\alpha f + (1 - \alpha) f\|_Y = \|\alpha F_1|_Y + (1 - \alpha) F_2|_Y\| \\ &\leq \|\alpha F_1 + (1 - \alpha) F_2\|_X \end{aligned}$$

showing that $\|\alpha F_1 + (1 - \alpha) F_2\|_X = \|f\|_Y$.

It follows that if $F_1, F_2 \in E_Y(f)$ then $\alpha F_1 + (1 - \alpha) F_2 \in E_Y(f)$ for $\alpha \in (0, 1)$, and $E_Y(f)$ is convex in the quasi-normed semilinear space $(SLip_0 X, \|\cdot\|_X)$.

Let (X, d) be a quasi-metric space, $x_0 \in X$ be fixed, and $Y \subset X$ with $x_0 \in Y$. Denote by

$$Y^\perp = \{g \in SLip_0 X : g|_Y = 0\}, \quad (6.2)$$

the *annihilator* of subset Y in $SLip_0 X$ and let

$$\rho : SLip_0 X \times SLip_0 X \rightarrow [0, \infty]$$

the (extended) quasi-metric defined by

$$\rho(f, g) = \sup_{\substack{x, y \in X \\ d(x, y) \neq 0}} \frac{((f - g)(x) - (f - g)(y)) \vee 0}{d(x, y)}$$

The following problem of best approximation (PBA) in the quasi-metric ρ is closely related to the problem of extension.

PBA: For $f \in SLip_0 X$ find a function $g_0 \in Y^\perp$ such that

$$\rho(f, g_0) = \inf \{\rho(f, g) : g \in Y^\perp\} = \rho(f, Y^\perp). \quad (6.3)$$

For $f \in SLip_0 X$ denote by

$$P_{Y^\perp}(f) = \{g_0 \in Y^\perp : \rho(f, g_0) = \rho(f, Y^\perp)\}, \quad (6.4)$$

the set of all elements of best approximation of f by elements of $Y^\perp$ in the quasi-metric ρ .

If $\text{card } P_Y(f) \geq 1$ then $Y^\perp$ is called f -proximal and if $\text{card } P_Y(f) = 1$, then $Y^\perp$ is called f -Chebyshevian.

Theorem 6.1. a) Let $f \in SLip_0 X$ and $F \in E_Y(f|_Y)$. If $f - F \in SLip X$, then $f - F \in P_{Y^\perp}(f)$ and

$$\rho(f, Y^\perp) = \|f|_Y\|_Y, \quad (6.5)$$

b) Let $f \in SLip_0 X$ and $g_0 \in P_Y(f)$. If $f - g_0 \in SLip_0 X$ then $f - g_0 \in E_Y(f|_Y)$.

Proof. a) Let be $f \in SLip_0 X$. Then $f|_Y \in SLip_0 Y$ and, by Theorem 2, the set $E_Y(f|_Y) \neq \emptyset$. If there exists $F \in E_Y(f|_Y)$ such that $f - F \in SLip_0 X$, then $f - F \in Y^\perp$ and for every $g \in Y^\perp$ we have

$$\begin{aligned} \|f|_Y\|_Y &= \|(f - g)|_Y\|_Y = \sup_{\substack{x, y \in Y \\ d(x, y) \neq 0}} \frac{((f - g)(x) - (f - g)(y)) \vee 0}{d(x, y)} \leq \\ &\leq \sup_{\substack{x, y \in X \\ d(x, y) \neq 0}} \frac{((f - g)(x) - (f - g)(y)) \vee 0}{d(x, y)} = \rho(f, g). \end{aligned}$$

Taking the infimum with respect to $g \in Y^\perp$ one gets

$$\|f|_Y\|_Y \leq \rho(f, Y^\perp).$$

On the other hand,

$$\rho(f, Y^\perp) \leq \rho(f, f - F) = \sup_{\substack{x, y \in X \\ d(x, y) \neq 0}} \frac{(F(x) - F(y)) \vee 0}{d(x, y)} = \|F\|_X = \|f|_Y\|_Y.$$

Consequently $\rho(f, Y^\perp) = \|f|_Y\|_Y$ and $Y^\perp$ is f -proximal.

b) Let be $g_0 \in Y^\perp$ such that $g_0 \in P_Y(f)$. If $f - g_0 \in SLip_0 X$ then

$$(f - g_0)|_Y = f|_Y,$$

and by a) one obtains

$$\begin{aligned} \|f|_Y\|_Y &= \rho(f, Y^\perp) = \\ &= \sup_{\substack{x, y \in X \\ d(x, y) \neq 0}} \frac{(f(f - g_0)(x) - (f - g_0)(y)) \vee 0}{d(x, y)} = \|f - g_0\|_X. \end{aligned}$$

It follows that $f - g_0 \in E_Y(f|_Y)$.

The Theorem is proved. $\square$

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