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ON THE STRUCTURE OF RESTRICTED GENERALIZED INFRAPOLYNOMIALS

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1. In the theory of infrapolynomials all characterisation theorems of the M. FEKETE type [4], are proved only for the infrapolynomials different from zero throughout the set to which they belong. In the algebraic case this restriction is not esential, because we can consider always a divisor different from zero, which is also an infrapolynomial. But when we consider the generalized infrapolynomials this restriction becomes essential, because in certain applications intervene the generalized infrapolynomials which have a finite number of zeros on the set to which the infrapolynomials belong. Hence, it is necessary to establish the characterisation theorems for like infrapolynomials too.

For the so called infraelements in the abstract spaces a similar characterisation was given by I. KOLUMBÁN [6].

In this paper we shall give a similar characterisation for a certain class of generalized infrapolynomials having a finite number of zeros on the set to which they belong. But the main results of this paper are the Theorems 4 and 5, which deal with the structure of the generalized infrapolynomials. This structure is similar with that given by G. CĂLUGĂREANU [1] to the Tchebycheff polynomial in the complex domain. Finally, we give an application of the Theorem 4 to the problem of preinterpolatory and next-to-interpolatory approximation by generalized polynomials.

Some theorems will be given without proofs, because these proofs may be found in other author's previous papers.

2. Let K be a bounded pointset of the complex plan, $\varphi_0, \varphi_1, \ldots, \varphi_n$, $n+1$ functions defined and continuous on K, and $\mathcal{P}(\alpha'; \varphi)$ the set of the generalized polynomials

$$p(z) = a_0\varphi_0(z) + a_1\varphi_1(z) + \ldots + a_n\varphi_n(z), \tag{1}$$

with coefficients $a_0, a_1, \ldots, a_n$ satisfying the conditions

$$\alpha_{i0}\, a_0 + \alpha_{i1}\, a_1 + \ldots + \alpha_{in}\, a_n = \alpha_i, \quad i = 0, 1, \ldots, r < n, \tag{2}$$

where

$$\alpha^r = \begin{pmatrix} \alpha_{00} \ldots \alpha_{0n}\alpha_0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_{r0} \ldots \alpha_{rn}\alpha_r \end{pmatrix}$$

is a given matrix of complex numbers, with

$$\operatorname{rang} ||\alpha_{ik}|| = r + 1, \quad \Sigma|\alpha_i| \neq 0.$$

Definition 1. *We say that $q(z) \in \mathcal{P}(\alpha^r ; \varphi)$ is an under polynomial of $p(z) \in \mathcal{P}(\alpha^r ; \varphi)$ on K, and we will use the notation $q(z) \in \mathfrak{U}(K, p ; \alpha^r)$, if*

I $\quad q(z) = 0$, *for* $z \in K$, $p(z) = 0$;

II $\quad |q(z)| < |p(z)|$, *for* $z \in K$, $p(z) \neq 0$.

Definition 2. *We say that $p(z) \in \mathcal{P}(\alpha^r ; \varphi)$ is a r-restricted generalized infrapolynomial on K, if $\mathfrak{U}(K, p ; \alpha^r) = \Lambda$, i.e. when $p(z)$ has no underpolynomial on K.*

The set of r-restricted generalized infrapolynomials on K will be denoted by $\mathcal{J}(K ; \alpha^r ; \varphi)$. Let K be a bounded pointset consisting at least $s+1$ points, and $\{\zeta_j\}_1^s$ s points of K. We denote by $\mathcal{P}(\zeta_j^r ; \varphi)$ the set of generalized polynomials (1), whose coefficients satisfy the conditions

$$a_0\, \varphi_0^{(h)}(\zeta_j) + a_1\, \varphi_1^{(h)}(\zeta_j) + \ldots + a_n\, \varphi_n^{(h)}(\zeta_j) = 0, \tag{3}$$

$$h = 0, 1, \ldots, r_j,\ j = 1, 2, \ldots, s;\ 1 \leqq r_1 + r_2 + \ldots + r_s \leqq n - r - s - 1$$

We denote by $\mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi)$ the set of the r-restricted generalized infrapolynomials, whose coefficients satisfy the conditions (3), and by $K_1 = K - \{\zeta_j\}_1^s$.

THEOREM 1. *A generalized polynomial $p(z) \in \mathcal{P}(\alpha^r ; \varphi)$ is a r-restricted generalized infrapolynomial on K, with zeros in ζ_j, $j = 1, 2, \ldots, s$, i.e. $p(z) \in \mathcal{J}(K ; \alpha^r ; \varphi)$, $p(\zeta_j) = 0$, $j = 1, 2, \ldots s$, $p(z) \neq 0$, $z \in K_1$, if and only if $p(z) \in \mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi)$, and $p(z) \neq 0$, $z \in K_1$.*

Necessity. Let $p(z) \in \mathcal{J}(K, \alpha^r ; \varphi)$ and $p(\zeta_j) = 0$, $j = 1, 2, \ldots, s$, $p(z) \neq 0$, $z \in K_1$, then it follows that $p(z) \in \mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi$. Indeed, if $\mathfrak{U}(K, p ; \alpha^r ; \zeta_j^r) \neq \Lambda$, then there exists a polynomial $q(z) \in \mathfrak{U}(K, p ; \alpha^r ; \zeta_j^r)$, $q(\zeta_j) = 0$, $j = 1, 2, \ldots, s$, i.e. $q(z) = 0$, for $z \in K$, $p(z) = 0$, and $|q(z)| < |p(z)|$, for $z \in K$, $p(z) \neq 0$. Therefore $q(z) \in \mathfrak{U}(K, p ; \alpha^r)$ which contradicts the fact that $p(z) \in \mathcal{J}(K ; \alpha^r ; \varphi)$.

Sufficiency. If $p(z) \in \mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi)$ and $p(z) \neq 0$, $z \in K_1$, then it follows that $p(z) \in \mathcal{J}(K ; \alpha^r, \varphi)$. Indeed, if $\mathfrak{U}(K, p ; \alpha^r) \neq \Lambda$, then there exists a polynomial $q(z) \in \mathfrak{U}(K, p ; \alpha^r)$, i.e. $q(z)$ satisfies (2) and (3), and because $p(z) \neq 0$, $z \in K_1$, it follows that

$$|q(z)| < |p(z)|, \ z \in K_1.$$

Therefore $q(z) \in \mathfrak{U}(K, p ; \alpha^r ; \zeta_j^r)$, which is a contradiction.

Definition 3. *We say that $M \subset K$ is a characteristic set for the r-restricted generalized infrapolynomial $p(z) \in \mathcal{J}(K ; \alpha^r ; \varphi)$, if $p(z) \in \mathcal{J}(M ; \alpha^r ; \varphi)$, and for any $M_1 \subset M$, $M_1 \neq M$, $p(z) \notin \mathcal{J}(M_1 ; \alpha^r ; \varphi)$.*

THEOREM 2. *If $p(z) \in \mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi)$, and $p(z) \neq 0$, $z \in K_1$, then any characteristic set M of $p(z)$ does not contain the points ζ_j, $j = 1, 2, \ldots, s$.*

Proof. Let us suppose on the contrary that $\zeta_1 \in M$, where M is a characteristic set for the r-restricted generalized infrapolynomial $p(z) \in \mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi)$. Then for $M_1 = M - \{\zeta_1\}$, we have $p(z) \in \mathcal{J}(M_1 ; \alpha^r ; \zeta_j^r ; \varphi)$.

Indeed if $\mathfrak{U}(M_1, p ; \alpha^r ; \zeta_j^r) \ni q(z)$, then it follows that $q(z)$ satisfies the conditions (2)--(3), and

$$|q(z)| < |p(z)|, \ z \in M - \{\zeta_j\}_1^s,$$

i.e. $q(z) \in \mathfrak{U}(M, p ; \alpha^r ; \zeta_j^r)$. But this contradicts the fact that M is a characteristic set for $p(z) \in \mathcal{J}(K ; \alpha^r ; \zeta_j^r ; \varphi)$. Therefore $M \not\ni \zeta$, $j = 1, 2, \ldots, s$.

By the Theorems 1 and 2, we have

Corollary 1. *If $p(z) \in \mathcal{J}(K ; \alpha^r ; \varphi)$, $p(\zeta_j) = 0$, $\zeta_j \in K$, $j = 1, 2, \ldots, s$, $p(z) \neq 0$, $z \in K_1$, then any characteristic set M of $p(z)$ does not contain the points ζ_j, j 1, 2, $\ldots$, s.*

It is known [4] that any algebraic infrapolynomial has a finite characteristic set when K is a compact pointset. Also when K is a compact pointset any r-restricted generalized infrapolynomial $p(z)$ has a finite characteristic set, if $p(z) \neq 0$, $z \in K$[10]. In the case when we consider the r-restricted

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generalized imfrapolynomials on a compact pontset K which is in the same times r-restricted generalized polylnomial of least deviation from zero on K, the existence of a finite characteristic set is always assured.

When K is not a closed pointset, then may be the existence of a finite characteristic set of a r-restricted generalized infrapolynomial belonging to K is not always assured. But we have the following

Corollary 2. *If $p(z) \in \mathcal{J}(K;\ \alpha^r;\ \varphi)$, $p(z) \not\equiv 0$, $z \in K_1$, and if there exists a compact set $C \subset K$ such that $p(z) \in \mathcal{J}(C;\ \alpha^r;\ \zeta^r_{jj};\ \varphi)$, $p(z) \not\equiv 0$, $z \in C$, then there exists a finite characteristic set for $p(z)$.*

Proof. If $p(z) \in \mathcal{J}(C;\ \alpha^r;\ \zeta^r_{jj};\ \varphi)$, $p(z) \not\equiv 0$, $z \in C$, then, because C is a compact set and $p(z) \not\equiv 0$, $z \in C$, there exists [10] a finite set $M = \{z_\nu\}_1^m \subset C$, $(1 \leqq m \leqq 2(n-r-s)+1)$ such that $p(z) \in \mathcal{J}(M;\ \alpha^r,\ \zeta^r_{jj};\ \varphi)$. By Theorem 1 it follows $p(z) \in \mathcal{J}(M;\ \alpha^r;\ \varphi)$, $p(z) \not\equiv 0$, $z \in M$, and it is obviously that M is a characteristic set for $p(z) \in \mathcal{J}(K;\ \alpha^r;\ \varphi)$.

Thus, the characterisation theorem of the M. Fekete type may be extended to the case when r-restricted generalized infrapolynomial $p(z)$ has some zeros on K, as follows.

THEOREM 3. *If $p(z) \in \mathcal{J}(K;\ \alpha^r;\ \varphi)$, $p(\zeta_j) = 0$, $\zeta_j \in K$, $j = 1, 2, \ldots, t$, $p(z) \not\equiv 0$, $z \in K_1$, and there exists a closed subset $C \subset K_1$, such that $p(z) \in \mathcal{J}(C;\ \alpha^r;\ \zeta^r_{jj},\ \varphi)$, then there exists m $(1 \leqq m \leqq 2(n-r-s)+1)$ points $z_\nu \in K_1$, the positive numbers $\lambda_\nu < 0$, with $\Sigma\lambda_\nu = 1$ and the complex numbers A_k, $k = 0, 1, \ldots r$, B_{jh}, $h = 0, 1, \ldots, r_j$, $j = 1, 2, \ldots, s$, such that*

$$(4)\qquad \sum_{\nu=1}^{m} \lambda_\nu \frac{\varphi_k(z_\nu)}{p(z_\nu)} = \sum_{i=0}^{r} \alpha_{ik} A_i + \sum_{j=0}^{s} \sum_{h=0}^{r_j} \varphi_k^{(h)}(\zeta_j)\, B_{jh}, \qquad k = 0, 1, \ldots, n;$$

$$(5)\qquad \sum_{i=0}^{r} \alpha_i A_i = 1,$$

and $M = \{z_\nu\}_1^m$ is a characteristic set for $p(z)$.

If a generalized polynomial $p(z) \in \mathcal{P}(\alpha^r;\ \varphi)$ satisfies (4) — (5), then $p(z) \in \mathcal{J}(K;\ \alpha^r;\ \zeta^r_{jj};\ \varphi)$.

Proof. If $p(z) \in \mathcal{J}(K;\ \alpha^r;\ \varphi)$, $p(\zeta_j) = 0$, $\zeta_j \in K$, $j = 1, 2, \ldots, s$, $p(z) \not\equiv 0$, $z \in K_1$, and there exists a compacts set $C \subset K_1$, such that $p(z) \in \mathcal{J}(C;\ \alpha^r,\ \zeta^r_{jj};\ \varphi)$, then $p(z)$ is a $r+s$-restricted generalized infrapolynomial on the compact set C, different from zero throughout on C, satisfying the linear restriction (2) — (3), and the Theorem 3 follows by Theorems 1 and 2 of our work [11].

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In the case when the r-restricted generalized infrapolynomial $p(z)$ has in ζ_j, $j = 1, 2, \ldots, s$, simple zeros, i.e. $p(z)$ satisfies the conditions

$$(3')\qquad a_0\varphi_0(\zeta_j) + a_1\varphi_1(\zeta_j) + \ldots + a_n\varphi_n(\zeta_j) = 0, \qquad j = 1, 2, \ldots, s,$$

then the conditions (4) of Theorem 3 get the form

$$(4')\qquad \sum_{\nu=1}^{m} \lambda_\nu \frac{\varphi_k(z_\nu)}{p(z_\nu)} = \sum_{i=0}^{r} \alpha_{ik} A_i + \sum_{j=0}^{s} \varphi_k(\zeta_j) B_j, \quad k = 0, 1, \ldots, n.$$

If $p(z)$ is different from zero troughout K, then in the right part of (4') absents the sum corresponding to $\varphi_k(\zeta_j)$, and we find again the result of our work [11], which also generalizes the result of M. FEKETE and J.L. WALSH [5], regarding the algebraic polynomials with restricted first k coefficients.

3. In the case of the restricted generalized infrapolynomials, different from zero troughout K, we may indicate their structure, puting in evidence the matrix and the points of the characteristic set.

First, let us give the following

Definition 4. *The system $\{\varphi_k\}_0^n$ of the continuous functions on K, forms a α_n^r — Tchebycheff system on K, if*

$$(6)\qquad D(z_1, \ldots, z_{n-r};\ \varphi;\ \alpha_n^r) = \begin{vmatrix} \alpha_{00} & \ldots & \alpha_{0n} \\ \ldots & \ldots & \ldots \\ \alpha_{r0} & \ldots & \alpha_{rn} \\ \varphi_0(z_1) & \ldots & \varphi_n(z_1) \\ \ldots & \ldots & \ldots \\ \varphi_0(z_{n-r}) & \ldots & \varphi_n(z_{n-r}) \end{vmatrix} \not\equiv 0$$

for all $z_j \in K$, $j = 1, 2, \ldots, n-r$.

It is obviously that a Tchebycheff system $\{\varphi_k\}_0^n$ on K is not always a α_n^r — Tchebycheff system on K, but any α_n^r — Tchebycheff system on K has the property that every generalized polynomial $p(z) \in \mathcal{P}(\alpha^r;\ \varphi)$, $p(z) \not\equiv 0$, has at most $n-r-1$ zeros on K. Also if $\{\varphi_k\}_0^n$ form a α_n^r — Tchebycheff system on K, then [10] every characteristic set for a r-restricted generalized infrapolynomial on K contains at least $n-r+1$ points, i.e. the inequalities

$$1 \leqq m \leqq 2(n-r)+1$$

in Theorem 3 can be replaced by the sharper estimate

$$n-r+1 \leqq m \leqq 2(n-r)+1.$$

Remark 1. When furthermore $\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z)$ are real valued troughout K, then these inequalities can be replaced by the equality $m = n - r + 1$.

THEOREM 4. [12]. *Let K be a compact pointset of the complex plane and suppose that $\{\varphi_k\}_0^n$ forms a α_n^r-Tchebycheff system on K. The generalized polynomial $p(z) \in \mathcal{P}(\alpha^r; \varphi)$, $p(z) \not\equiv 0$, $z \in K$ is a r-restricted generalized infrapolynomial on K, i.e. $p(z) \in \mathcal{J}(K; \alpha^r; \varphi)$, if and only if there exists a set $M = \{z_\nu\}_1^m \subset K$ $(n - r + 1 \leqq m \leqq 2(n - r) + 1$, and the positive constants d_ν, such that*

$$(7) \qquad p(z) = \frac{(-1)^r \Sigma d_{\nu_1} \ldots d_{\nu_{n-r}} \overline{D}(z_{\nu_1}, \ldots, z_{\nu_{n-r}}; \varphi; \alpha_n^r) V(z, z_{\nu_1}, \ldots, z_{\nu_{n-r}}; \varphi; \alpha^r)}{\Sigma d_{\nu_1} \ldots d_{\nu_{n-r}} |D(z_{\nu_1}, \ldots, z_{\nu_{n-r}}; \varphi; \alpha_n^r)|^2},$$

where

$$V(z, x_1, \ldots, x_{n-r}; \varphi; \alpha^r) = \begin{vmatrix} \alpha_0 & \alpha_{00} & \ldots & \alpha_{0n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_r & \alpha_{r0} & \ldots & \alpha_{rn} \\ 0 & \varphi_0(z) & \ldots & \varphi_n(z) \\ 0 & \varphi_0(x_1) & \ldots & \varphi_n(x_1) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \varphi_0(x_{n-r}) & \ldots & \varphi_n(x_{n-r}) \end{vmatrix}$$

$\overline{D}(x_1, \ldots, x_{n-r}; \varphi; \alpha_n^r)$ being the conjugate of $D(x_1, \ldots, x_{n-r}; \varphi; \alpha_n^r)$ and Σ is taken for all ν_k form 1 to m.

By virtue of Theorem 3, the r-restricted generalized infrapolynomials which have some zeros on K, have a similar structure. For simplicity, we state the theorem in the case of the simple zeros, i.e. for the case when are statisfied the conditions (3'). We denote by $\mathcal{J}(K; \alpha^r; \zeta_j; \varphi)$ the set of the generalized infrapolynomials whose coefficients satisfy the condidions (2)–(3').

THEOREM 5. *Let $\{\varphi_k\}_0^n$ be a α_n^r-Tchebycheff system on K and $p(z) \in \mathcal{J}(K; \alpha^r\}; \zeta_j; \varphi)$, $p(z) \not\equiv 0$, $z \in K_1$. If there exists a closed set $C \subset K_1$, such that $p(z) \in \mathcal{J}(C; \alpha^r; \zeta_j; \varphi)$, then there exists m $(n - r - s + 1 \leqq m \leqq 2(n - r - s) + 1)$ points $z_\nu \in C$ and the positive constants d_ν, such that*

$$(8) \qquad p(z) =$$

$$= \frac{(-1)^{r+s} \Sigma d_{\nu_1} \ldots d_{\nu_{n-r-s}} \overline{D}(\zeta_1, \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-r-s}}; \varphi; \alpha_n^r) V(\zeta_1, \ldots, \zeta_s, z, z_{\nu_1}, \ldots, z_{\nu_{n-r-s}}; \varphi; \alpha^r)}{\Sigma' d_{\nu_1} \ldots d_{\nu_{n-r-s}} |D(\zeta_1 \ldots \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-r-s}}; \varphi; \alpha_n^r)|^2},$$

If a generalized polynomial $p(z) \in \mathcal{P}(\alpha^r; \varphi)$ satisfies the equality (8), *then $p(z) \in \mathcal{J}(K; \alpha^r; \zeta_j; \varphi)$.*

Definition 5. *The generalized polynomial*

$$(9) \qquad q(z) = b_1\varphi_1(z) + b_2\varphi_2(z) + \ldots + b_n\varphi_n(z)$$

is a generalized juxtapolynomial to φ_0 on K, if the generalized infrapolynomial

$$(10) \qquad p(z) = -\varphi_0(z) + q(z)$$

is a o-restricted generalized infrapolynomial, where $\alpha_{00} = -1$, $\alpha_0 = 1$, $\alpha_{0j} = 0$, $j = 1, 2, \ldots, n$.

By the Theorems 4 and 5 it follows immediatly the structure of the generalized juxtapolynomial to a function continuous on K.

Corollary 3. *Under the conditions of Theorem 4 for K and $\{\varphi_k\}_0^n$, a generalized polynomial $q(z)$ of the form* (9), *$q(z) \not\equiv \varphi_0(z)$, $z \in K$, is a generalized juxtapolynomial to φ_0 on K, if and only if, there exists a set $M = \{z_\nu\}_1^m \subset K$, $(n + 1 \leqq m \leqq 2n + 1)$, and the positive constants $\Lambda_{\nu_1 \ldots \nu_n}$, with $\Sigma\Lambda_{\nu_1 \ldots \nu_n} = 1$, such that*

$$(10) \qquad q(z) = \Sigma\Lambda_{\nu_1 \ldots \nu_n} L^\circ(z_{\nu_1}, \ldots, z_{\nu_n}; \varphi|z),$$

where $L^\circ(x_1, x_2, \ldots, x_n; \varphi|z)$ is the generalized Lagrange's interpolatory polynomial of the form (9) *on the knots $x_1, x_2, \ldots, x_n$ of φ_0.*

Corollary 4. *Under the conditions of Theorem 5 for K and $\{\varphi_k\}_0^n$, a generalized polynomial $q(z)$ of the form* (9), *$q(\zeta_j) = \varphi_0(\zeta_j)$, $\zeta_j \in K$, $j = 1, 2, \ldots, s$, $q(z) \not\equiv \varphi_0(z)$, $z \in K_1$, is a generalized juxtapolynomial on K_1, if and only if, there exits a set $M = \{z_\nu\}_1^m \subset K_1$, $(n - s + 1 \leqq m \leqq 2(n - s) + 1)$, and the positive constants $\Lambda_{\nu_1 \ldots \nu_{n-s}}$ with $\Sigma\Lambda_{\nu_1 \ldots \nu_{n-s}} = 1$, such that*

$$(11) \qquad q(z) = \Sigma\Lambda_{\nu_1 \ldots \nu_{n-s}} L^\circ(\zeta_1 \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi|z).$$

Indeed, in the case when $r = 0$, $\alpha_{00} = -1$, $\alpha_0 = 1$, $\alpha_{0k} = 0$, $k = 1, 2, \ldots, n$, the structure (8) of the o-restricted generalized infrapolynomial $p(z) = -\varphi_0(z) + q(z)$ can be written in the form

$$(12) \qquad -\varphi_0(z) + q(z) =$$

$$= \frac{(-1)^{s+1} \Sigma d_{\nu_1} \ldots d_{\nu_{n-1}} \overline{D}(\zeta_1, \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi) V(\zeta_1, \ldots, \zeta_s, z, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi)}{\Sigma d_{\nu_1} \ldots d_{\nu_{n-s}} |D(\zeta_1, \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi)|^2},$$

where

$$V(\zeta_1, \ldots, \zeta_s, x_1, \ldots, x_{n-s}; \varphi) = \begin{vmatrix} \varphi_0(\zeta_1) & \ldots & \varphi_n(\zeta_1) \\ \ldots & \ldots & \ldots \\ \varphi_0(\zeta_s) & \ldots & \varphi_n(\zeta_s) \\ \varphi_0(z) & \ldots & \varphi_n(z) \\ \varphi_0(x_1) & \ldots & \varphi_n(x_1) \\ \ldots & \ldots & \ldots \\ \varphi_0(x_{n-s}) & \ldots & \varphi_n(x_{n-s}) \end{vmatrix}$$

and

$$D(x_1, x_2 \ldots, x_n; \varphi) = \begin{vmatrix} \varphi_1(x_1) & \ldots & \varphi_n(x_1) \\ \varphi_2(x_2) & \ldots & \varphi_n(x_2) \\ \ldots & \ldots & \ldots \\ \varphi_n(x_n) & \ldots & \varphi_n(x_n) \end{vmatrix}.$$

But because

$$\frac{V(\zeta_1, \ldots, \zeta_s, z, x_1, \ldots, x_{n-s}; \varphi)}{D(\zeta_1, \ldots, \zeta_s, x_1, \ldots, x_{n-s}; \varphi)} = (-1)^s [\varphi_0(z) - L^0(\zeta_1, \ldots, \zeta_s, x_1, \ldots, x_{n-s}; \varphi|z)],$$

by (12), puting

$$\Lambda_{\nu_1 \ldots \nu_{n-s}} = \frac{d_{\nu_1} \ldots d_{\nu_{n-s}} |D(\zeta_1, \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi)|^2}{\Sigma d_{\nu_1} \ldots d_{\nu_{n-s}} |D(\zeta_1, \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi)|^2},$$

we obtain

$$-\varphi_0(z) + q(z) = -\varphi_0(z) + \Sigma \Lambda_{\nu_1 \ldots \nu_{n-s}} L^0(\zeta_1 \ldots, \zeta_s, z_{\nu_1}, \ldots, z_{\nu_{n-s}}; \varphi|z),$$

i.e. $q(z)$ is given by (11).

When $q(z) \not\equiv \varphi_0(z)$, $z \in K$, then $s = 0$, and by (11) it follows

$$q(z) = \Sigma \Lambda_{\nu_1 \ldots \nu_n} L^0(z_{\nu_1}, \ldots z_{\nu_n}; \varphi|z),$$

i.e. (10) is proved.

Remark 2. When the functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ are real valued throughout K, then by Remark 1 it follows that M consists from $m = n + 1$

points, and therefore the structure of the generalized juxtapolynomial $q(z)$ to $\varphi_0(z)$ on K, $q(z) \not\equiv \varphi_0(z)$, $z \in K$, is the following

$$q(z) = \sum_{k=1}^{n+1} \Lambda_k L^0(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_{n+1}; \varphi|z). \tag{10'}$$

From (7) it follows in special the structure of the generalized infrapolynomial with respect to the coefficients $a_{n_0}, a_{n_1}, \ldots, a_{n_r}$ $(0 \leqq n_0 < n_1 < \ldots < n_r \leqq n;\ 0 \leqq r < n)$. Thus, if we take the matrix α^r as follow

$$\alpha_j = A_{n_i},\ \alpha_{in_i} = 1,\ i = 0, 1, \ldots, r, \tag{13}$$

and the rest $\alpha_{jk} = 0$, we have

$$D(x_1, x_2, \ldots, x_{n-r}; \varphi, \alpha_n^r) = (-1)^{h_r} U \begin{pmatrix} x_1, x_2, \ldots, x_{n-r} \\ \varphi_{m_1}, \varphi_{m_2}, \ldots, \varphi_{m_{n-r}} \end{pmatrix},$$

where

$$h_r = 1 + 2 + \ldots + r + n_0 + n_1 + \ldots + n_r$$

and

$$U \begin{pmatrix} x_1, x_2, \ldots, x_{n-r} \\ \varphi_{m_1}, \varphi_{m_2}, \ldots, \varphi_{m_{n-r}} \end{pmatrix} = \begin{vmatrix} \varphi_{m_1}(x_1) & \ldots & \varphi_{m_{n-r}}(x_1) \\ \varphi_{m_1}(x_2) & \ldots & \varphi_{m_{n-r}}(x_2) \\ \ldots & \ldots & \ldots \\ \varphi_{m_1}(x_{n-r}) & \ldots & \varphi_{m_{n-r}}(x_{n-r}) \end{vmatrix},$$

$$m_k \in N - \sigma,\ N = \{0, 1, 2, \ldots, n\},\ \sigma = \{n_0, n_1, \ldots, n_r\}.$$

Similarly, we have

$$V(z, x_1, \ldots, x_{n-r}; \varphi; \alpha^r) = \sum_{j=0}^{r} (-1)^{h_r - r} A_{n_j} U \begin{pmatrix} z, x_1 \ldots, x_{n-r} \\ \varphi_{n_j}, \varphi_{m_1}, \ldots, \varphi_{m_{n-r}} \end{pmatrix}.$$

Because

$$\frac{U \begin{pmatrix} z, x_1, \ldots, x_{n-r} \\ \varphi_{n_j}, \varphi_{m_1}, \ldots, \varphi_{m_{n-r}} \end{pmatrix}}{U \begin{pmatrix} x_1, x_2, \ldots, x_{n-r} \\ \varphi_{m_1}, \varphi_{m_2}, \ldots, \varphi_{m_{n-r}} \end{pmatrix}} = \varphi_{n_j}(z) - L \begin{pmatrix} x_1, x_2, \ldots, x_{n-r} \\ \varphi_{m_1}, \varphi_{m_2}, \ldots, \varphi_{m_{n-r}} \end{pmatrix} ; \varphi_{n_j} | z \Bigg)$$

where

$$L\left(\begin{matrix}x_1, x_2, \ldots, x_{n-r}\\ \varphi_{m_1}, \varphi_{m_2}, \ldots, \varphi_{m_{n-r}}\end{matrix}; \varphi_{n_j} \mid z\right) = l_1 \varphi_{m_1}(z) + l_2 \varphi_{m_2}(z) + \ldots + l_{m-r} \varphi_{m_{n-r}}(z)$$

is the generalized interpolatory polynomial of the function φ_{n_j} on the knots $x_1, x_2, \ldots, x_{n-r}$, from (7) it follows immediatly

$$(14) \quad p(z) = \Sigma \Lambda_{\nu_1 \ldots \nu_{n-r}} A_{n_j}\left[\varphi_{n_j}(z) - L\left(\begin{matrix}x_1, & x_2, & \ldots, & x_{n-r}\\ \varphi_{m_1}, & \varphi_{m_2}, & \ldots, & \varphi_{m_{n-r}}\end{matrix}; \varphi_{n_j} \mid z\right)\right].$$

Now if we denote by

$$\pi_{n_0, \ldots, n_r}(z) = A_{n_0} \varphi_{n_0}(z) + A_{n_1} \varphi_{n_1}(z) + \ldots + A_{n_r} \varphi_{n_r}(z)$$

the given generalized polynomial, then for (13) it follows the following

Corollary 5. *Under the conditions of Theorem 4 for K and $\{\varphi_k\}_0^n$ a generalized polynomial $p(z)$, with prescribed coefficients $A_{n_0}, A_{n_1}, \ldots, A_{n_r}$, $p(z) \not\equiv 0$, $z \in K$, is a generalized infrapolynomial with respect to the coefficients $(a_{n_0}, a_{n_1}, \ldots, a_{n_r})$ on K, if and only if there exists a set $M = \{z_\nu\}_1^m \subset K$, $(n - r + 1 \leqq m \leqq 2(n - r) + 1$, and the positive constants $\Lambda_{\nu_1 \ldots \nu_{n-r}}$ with $\Sigma \Lambda_{\nu_1 \ldots \nu_{n-r}} = 1$, such that*

$$(15) \quad p(z) = \pi_{n_0, n_1, \ldots, n_r}(z) - \Sigma \Lambda_{\nu_1 \ldots \nu_{n-r}} L\left(\begin{matrix}z_{\nu_1}, & \ldots, & z_{\nu_{n-r}}\\ \varphi_{m_1}, & \ldots, & \varphi_{m_{n-r}}\end{matrix}; \pi \mid z\right).$$

In the algebraic case

$$\pi_{n_0, n_1, \ldots, n_r}(z) = A_{n_0} z^{n_0} + A_{n_1} z^{n_1} + \ldots A_{n_r} z^{n_r}$$

and $\{z^k\}_0^n$ forms a α_n^r-Tchebycheff system on K, with α_{ik} given in (13), wher $\{z^{m_i}\}_1^{n-r}$ forms a Tchebycheff system on K, i.e. when $m_1 = 0$, $m_2 = 1, \ldots, m_{n-r} = n - r - 1$.

A simple n-sequance is called (after O. SHISHA [16]) a sequance $\sigma = (n_0, n_1, \ldots, n_r)$ if $N - \sigma = (m_1, m_2, \ldots, m_{n-r})$ is „connected", i.e. $N - \sigma$ is of the form $(k, k + 1, \ldots, k + n - r)$, $(0 \leqq k \leqq r)$.

If $(n_0, n_1, \ldots, n_r)$ is a simple n-sequance, then $\{z^{m_i}\}_1^{n-r}$ is a Tchebycheff system on any set K which does not contain the origin 0. Thus the conditions of the Theorem 4 will be always satisfied if $k = 0$, and for the sets $K \not\ni 0$, if $k \neq 0$.

In this case

$$D(x_1, x_2, \ldots, x_{n-r}; \alpha_u^r) = \pm x_1^k x_2^k \ldots x_{n-r}^k V(x_1, x_2, \ldots, x_{n-r}),$$

where $V(x_1, x_2, \ldots, x_{n-r})$ is a Vandermonde determinant of the numbers $x_1, x_2, \ldots, x_{n-r}$, and the interpolation polynomial,

$$L\left(\begin{matrix}x_1, x_2, \ldots, x_{n-r}\\ z^k, z^{k+1}, \ldots, z^{n+k-r}\end{matrix}; \pi \mid z\right) = z^k \sum_{j=0}^{n-r} \frac{g(z)}{g'(z_j)} \cdot \frac{\pi(z_j)}{z_j^k (z - z_j)},$$

where

$$g(z) = (z - z_1)(z - z_2) \ldots (z - z_{n-r}).$$

The structure of the infrapolynomials with respect to the coefficients $(A_{n_0}, A_{n_1}, \ldots, A_{n_r})$ where $\delta = (n_0, n_1, \ldots, n_r)$ is simple n-sequence, is given by the following.

Corollary 6. *Let σ be a simple n-sequance and K a compact set. In case when $k \neq 0$, assume that $0 \notin K$. A polynomial*

$$P(z) = \pi_{k, \ldots, n+k-r}(z) + a_k z^k + \ldots + a_{n+k-r} z^{n+k-r},$$

where

$$\pi_{k, \ldots, n+k-r}(z) = A_0 + A_1 z + \ldots + A_{k-1} z^{k-1} + A_{n+k-r+1} z^{n+k-r+1} + \ldots + A_n z^n,$$

$P(z) \not\equiv 0$, $z \subset K$, is a infrapolynomial on K with respect to the coefficients $(A_0, A_1, \ldots, A_k, A_{n+k-r+1}, \ldots, A_n)$ if and only if there exists the set $M = \{z_\nu\}_1^m \subset K$, $(n - r + 1 \leqq m \leqq 2(n - r) + 1)$, and the positive constants $\Lambda_{\nu_1 \ldots \nu_{n-r}}$, with $\Sigma \Lambda_{\nu_1 \ldots \nu_{n-r}} = 1$, such that

$$(15) \quad P(z) = \pi_{k, \ldots, n+k-r}(z) - z^k \Sigma \Lambda_{\nu_1 \ldots \nu_{n-r}} \sum_{j=1}^{n-r} \frac{g_{\nu_1 \ldots \nu_{n-r}}(z)}{g'_{\nu_1 \ldots \nu_{n-r}}(z_{\nu_j})} \cdot \frac{\pi(z_{\nu_j})}{z_{\nu_j}^k (z - z_{\nu_j})}$$

where

$$g_{\nu_1 \ldots \nu_{n-q}}(z) = (z - z_{\nu_1})(z - z_{\nu_2}) \ldots (z - z_{\nu_{n-r}}).$$

In the case $r = 0$, $A_n = 1$, we have for the classic infrapolynomials the structure

$$P(z) = z^n - \Sigma \Lambda_{\nu_1 \ldots \nu_n} \sum_{j=1}^{n} \frac{g_{\nu_1 \ldots \nu_n}(z)}{g'_{\nu_1 \ldots \nu_n}(z_{\nu_j})} \cdot \frac{z_{\nu_j}^n}{z - z_{\nu_j}}$$

which may be also written under the form

$$P(z) = \Sigma\Lambda_{\nu_1 \dots \nu_n}(z - z_{\nu_1})(z - z_{\nu_2}) \dots (z - z_{\nu_n}). \tag{16}$$

When $P(z)$ is a Tchebycheff polynomials on K, the form (16) was established, but using another method, by G. CĂLUGĂREANU [1].

Remark 3. If for the classic infrapolynomial $P(z)$ on K we have $P(\zeta_j) = 0$, $\zeta_j \in K$, $j = 1, 2, \dots, s$ then using the Theorem 5, it follows immediatly that

$$P(z) =$$

$$= (-1)^s(z-\zeta_1)(z-\zeta_2)\dots(z-\zeta_s)\,\Sigma\Lambda_{\nu_1 \dots \nu_{n-s}}\,(z - z_{\nu_1})(z-z_{\nu_2}) \dots (z - z_{\nu_{n-s}}),$$

where

$$\Lambda_{\nu_1 \dots \nu_{n-s}} > 0,\ \Sigma\Lambda_{\nu_1 \dots \nu_{n-s}} = 1.$$

Therefore,

$$P(z) = P_s(z)\,P_{n-s}(z),$$

where $P_s(z)$ and $P_{n-s}(z)$ are the infrapolynomials of degrees s and $n-s$ respectively, $P_{n-s}(z) \not\equiv 0$, $z \in K$. This result is due to M. FEKETE and J. L. WALSH [5].

4. Finally we shall give an application of the r-restricted generalized infrapolynomials to the interpolation theory.

Let K be again a comact set in the complex plan, containing at least $n+1$ points, and $\zeta_j \in K$, $j = 0, 1, \dots, s$. We denote by $\mathcal{P}_s(\zeta_j, w_j; \varphi)$ the set of the generalized polynomials (1), whose coefficients satisfy the conditions

$$a_0\varphi_0(\zeta_j) + a_1\varphi_1(\zeta_j) + \dots + a_n\varphi_n(\zeta_j) = w_j,\ j = 0, 1, \dots s. \tag{17}$$

Definition 6. *A generalized polynomial $p(z)$ is called generalized preinterpolatory infrapolynomial on K of the order s, if $p(z)$ satisfies the conditions* (17), *and if it is a generalized infrapolynomial on K.*

We denote by $\mathcal{J}_s(K; \zeta_j; w_j; \varphi)$ the set of the generalized preinterpolatory infrapolynomial on K.

If $\{\varphi_k\}_0^n$ is a Tchebycheff system on K, then the set $\mathcal{J}_n(K; \zeta_j, w_j; \varphi)$ contains only a single element: the Lagrange interpolatory polynomial on the knots $\zeta_0, \zeta_1, \dots, \zeta_n$, which is unique.

Remark 4. If $\{\varphi_k\}_0^n$ forms a Tchebycheff system on K, then $\{\varphi_k\}_0^n$ forms a α_n'-Tchebycheff system on K too, with

$$\alpha_{ik} = \varphi_k(\zeta_i),\ i = 0, 1, \dots, s;\ k = 0, 1, \dots, n. \tag{18}$$

From Remark 4 it follows that the conditions of Theorem 4 are satisfied, and we may indicate the similar structure for the generalized preinterpolatory infrapolynomials. Thus we have

THOEREM 6. *Let K be a compact set containing at least $n+1$ points and assume that $\{\varphi_k\}_0^n$ forms a Tchebycheff system on K. A generalized polynomial $p(z) \in \mathfrak{P}_s(\zeta_j; w_j; \varphi)$, $p(z) \not\equiv 0$, $z \in K_1 = K - \{\zeta_j\}_0^s$, is a generalized preinterpolatory infrapolynomial of the order s on K, i.e. $p(z) \in \in \mathcal{J}_s(K; \zeta_j; w_j; \varphi)$, if and only if there exists a subset $M = \{z_\nu\}_1^m \subset K_1$, $(n - s + 1 \leq m \leq 2(n-s) + 1)$ and the positive constants d_ν, such that*

$$p(z) = \tag{19}$$

$$= \frac{(-1)^s \Sigma d_{\nu_1} \dots d_{\nu_{n-s}}\, U(\zeta_0, \dots, \zeta_s, z_{\nu_1}, \dots, z_{\nu_{n-s}}; \varphi)\, V(\zeta_0, \dots, \zeta_s\, z, z_{\nu_1}, \dots, z_{\nu_{n-s}}; \varphi)}{\Sigma d_{\nu_1} \dots d_{\nu_{n-s}} \mid U(\zeta_0, \dots, \zeta_s, z_{\nu_1}, \dots, z_{\nu_{n-s}}; \varphi)|^2},$$

where

$$V(\zeta_0, \dots, \zeta_s, z, x_1, \dots, x_{n-s}; \varphi) = \begin{vmatrix} w_0\varphi_0(\zeta_0) & \dots \varphi_n(\zeta_0) \\ \dots\dots\dots\dots\dots \\ w_s\varphi_0(\zeta_s) & \dots \varphi_n(\zeta_s) \\ 0\ \varphi_0(z) & \dots \varphi_n(z) \\ 0\ \varphi_0(z_1) & \dots \varphi_n(x_1) \\ \dots\dots\dots\dots\dots \\ 0\ \varphi_0(x_{n-s}) & \dots \varphi_n(x_{n-s}) \end{vmatrix}.$$

If we introduce the special generalized interpolatory polynomial $l(\zeta_0, \dots, \zeta_s, x_1, \dots, x_{n-s}; \varphi|z)$, which satisfies the conditions

$$l(\zeta_0, \dots, \zeta_s, x_1, \dots, x_{n-s}; \varphi|\zeta_j) = w_j,\ j = 0, 1, \dots, s, \tag{20}$$

$$l(\zeta_0, \dots, x_1, \dots x_{n-s}; \varphi|x_k) = 0,\ k = 1, 2, \dots, n-s, \tag{21}$$

then the generalized preinterpolatory infrapolynomial on K of the order s, given in (19), can be write on the form

$$p(z) = \Sigma\Lambda_{\nu_1 \dots \nu_{n-k}}\, l(\zeta_0, \dots, \zeta_s, z_{\nu_1}, \dots, z_{\nu_{n-s}}; \varphi|z), \tag{19'}$$

where

$$\Lambda_{\nu_1 \dots \nu_{n-s}} > 0, \quad \Sigma \Lambda_{\nu_1 \dots \nu_{n-s}} = 1.$$

1° If $s = n$, then form (19′) it follows

$$p(z) = l(\zeta_0, \dots, \zeta_n ; \varphi|z),$$

i.e. Lagrange's generalized interpolatory polynomial on $\zeta_0, \zeta_1, \dots, \zeta_n$.

2° If $s = n - 1$, then we obtain

$$p(z) = \sum_{\nu=1}^{m} \Lambda_\nu l(\zeta_0, \zeta_1, \dots, \zeta_{n-1}, z_\nu ; \varphi|z), \tag{22}$$

where $2 \leqq m \leqq 3$. This is a generalization of a result given by T. S. MOTZKIN and A. SHARMA [15] for a so-called nex-to-interpolatory polynomial.

Remark. If the generalized preinsterpolatory infrapolynomial has the zeros on K_1, then by using the Theorem 5 it may be easly indicated a similar structure for this preinterpolatory infrapolynomial too.

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