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QUASIBOUNDED MAPPINGS AND GENERALIZED CONTRACTIONS

by

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In the fixed point theory there are several theorems where the condition imposed to the function for which the existence of the fixed point is proved is to be a contraction or to be quasibounded. In this paper we try to give an answer to a problem proposed by I.A.Rus [2], that is under what circumstances a generalized contraction is quasibounded and has the quasinorm $|f| < 1$.

Definition 1. A function $\varphi : R_+^2 \to R_+$ is a strict comparison function if

(i) $(r \leq s) \Rightarrow \varphi(r) \leq \varphi(s)$

(ii) $(r \in R_+, r > 0) \Rightarrow (\varphi(r,r,r,r,r) < r)$

(iii) $r - \varphi(r,r,r,r,r) \to +\infty$, when $r \to +\infty$

(iv) φ is continuous.

Definition 2. Let (X,d) be a metric space. A function $f : X \to X$ is called φ-contraction if there is a strict comparison function $\varphi : R_+^2 \to R_+$ such that

$$d(f(x),f(y)) \leq \varphi(d(x,y),d(x,f(x)),d(y,f(y)),d(x,f(y)),d(y,f(x))), \quad \forall x,y \in X.$$

Definition 3. Let $(X,|.|)$ be a linear normed space. A function $f : X \to X$ is called quasibounded if there are $a, b \geq 0$ such that $|f(x)| \leq a|x| + b$, $\forall x \in X$. The quasinorm of f, denoted by $|f|$, is by definition $|f| = \inf \{a \in R_+ \mid |f(x)| \leq a|x| + b, \forall x \in X\}$.

Definition 4. Let $(X,|.|)$ be a linear normed space and $f :$

$: X \to X$ a quasibounded function. The function f is called normcontractive if $|f| < 1$.

In the following $(X, |.|)$ will be a linear normed space. The different types of contractions that will appear are to be founded in the paper of B.E.Rhoades[1].

Proposition 1. The generalized contractions $f : X \to X$ which satisfy one of the following conditions are normcontractive

(1) $|f(x)-f(y)| \leq a|x-y| + b|x-f(x)| + c|y-f(y)| + d|x-f(y)| + e|y-f(x)|$, $\forall x,y \in X$,

where $0 \leq a,b,c,d,e$; $a+b+c+d+e < 1$ (Hardy, Rogers),

(2) for any $x,y, \in X$, at least one of the following relations is satisfied

(i) $|f(x)-f(y)| \leq a|x-y|$

(ii) $|f(x)-f(y)| \leq b(|x-f(x)| + |y-f(y)|)$

(iii) $|f(x)-f(y)| \leq c(|x-f(y)| + |y-f(x)|)$

where $0 \leq a < 1$, $0 \leq b,c < 1/2$ (Zamfirescu).

Proof.

Let f satisfy (1). Then for y=0 we obtain

$|f(x)| \leq |f(0)| + |f(x)-f(0)| \leq |f(0)| + a|x| + b|x| + b|f(x)| + c|f(0)| + d|x| + d|f(0)| + e|f(x)|$ and finally

$|f(x)| \leq \frac{a+b+d}{1-b-e}|x| + \frac{1+c+d}{1-b-e}|f(0)|$, $\forall x \in X$.

If we permute x and y in (1) we have

$|f(x)| \leq \frac{a+c+e}{1-c-d}|x| + \frac{1+b+e}{1-c-d}|f(0)|$, $\forall x \in X$.

From $a+b+c+d+e < 1$ we obtain ($a+2b+d+e < 1$ or $a+2c+d+e < 1$), that is ($\frac{a+b+d}{1-b-e} < 1$ or $\frac{a+c+e}{1-c-d} < 1$). Consequently, $|f| < 1$.

When f satisfies (2) we obtain

$|f(x)| \leq |f(0)| + a.\max\{|x-f(0)|, |f(x)|\}$, $\forall x \in X$, hence

$|f(x)| \leq a|x| + \max\{1+a, \frac{1}{1-a}\}\cdot|f(0)| = a|x| + \frac{1}{1-a}|f(0)|$, $\forall x \in X$ and then $|f| \leq a < 1$.

In the case of the condition (3) we have similarly

$|f(x)| \leq \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}\cdot|x| + \max\{1, \frac{1+b}{1-b}, \frac{1+c}{1-c}\}|f(0)|$.

It follows that $|f| \leq \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\} < 1$.

Remark 1. In the case that in (1) some of the numbers a, b, c, d, e are null, we obtain some well-known generalized contractions (Banach, Kannan, Reich, Rus, Chatterjea, Ćirić), all of them being then normcontractive.

The following type of contractions is due to Bianchini

(4) $|f(x)-f(y)| \leq a.\max\{|x-f(x)|, |y-f(y)|\}$, $\forall x,y \in X$,

where $0 \leq a < 1$.

Ćirić has introduced another type of contractions

(5) $|f(x)-f(y)| \leq a.\max\{|x-y|, |x-f(x)|, |y-f(y)|, |x-f(y)|, |y-f(x)|\}$, $\forall x,y \in X$,

where $0 \leq a < 1$.

In these cases we prove that the contractions are normcontractive only if $0 \leq a < 1/2$.

Proposition 2. The generalized contractions that satisfy the condition (4) or (5) with $0 \leq a < 1/2$ are normcontractive.

Proof.

Let f satisfy (4). For y=0 we obtain from (4)

$|f(x)| \leq \frac{a}{1-a}|x| + \max\{\frac{1}{1-a}, 1+a\}\cdot|f(0)| = \frac{a}{1-a}|x| + \frac{1}{1-a}|f(0)|$, $\forall x \in X$,

hence $|f| \leq \frac{a}{1-a} < 1$ for $a < 1/2$.

When f satisfies (5) we have

$|f(x)| \leq \frac{a}{1-a}|x| + \frac{1}{1-a}|f(0)|$, $\forall x \in X$,

so $|f| \leq \frac{a}{1-a} < 1$ for $a < 1/2$.

Remark 2. There are generalized contractions satisfying (4) or (5) with $1/2 \leq a < 1$ which are not normcontractive, as the following example shows.

Example 1. Let $f : R \to R$ given by $f(x) = \begin{cases} -kx, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$,

$k \in [1, +\infty)$. f verifies the conditions (4) and (5) with $a = k/(k+1) \geq 1/2$, but $|f| = k \geq 1$. The case $a = 1/2$ is obtained for $k = 1$.

In conclusion, even if many generalized contractions are normcontractive, there are some which have not this property; there are also normcontractive mappings which are not generalized contractions.

Example 2. Let $f : R \to R$ given by $f(x) = \begin{cases} 0, & x \leq 0 \\ kx, & 0 < x < a, \\ ka, & x \geq a \end{cases}$

where $a > 0$ and $k \geq 1$, which is clearly normcontractive. We suppose that there is a strict comparison function φ with respect to which f is a φ-contraction.

Let x=o and y=a. We have

$ka = |f(x)-f(y)| \leq \varphi(a, 0, ka-a, ka, a) \leq \varphi(ka, \ldots, ka) < ka$, contradiction. It follows that f is not a generalized contraction.

REFERENCES

1. B.E.Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290

2. I.A.Rus, Probleme actuale în analiza neliniară, Lucrările Seminarului T.Angheluţă, Inst. Polit. Cluj-Napoca 1983.