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ON THE CONTINUITY OF POINT-TO-SET MAPPINGS

by

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We consider in this note point-to-set mappings with values in a metric space.

Let (X,d) be a metric space, where we denote by $\mathcal{P}(X)$ the family of the nonempty subsets of X and

$d(x,A) = \inf\{d(x,a) : a \in A\}$, $x \in X$, $A \in \mathcal{P}(X)$

$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$, $A,B \in \mathcal{P}(X)$

$B(A,\varepsilon) = \{x \in X : d(x,A) < \varepsilon\}$, $A \in \mathcal{P}(X)$, $\varepsilon > 0$; $B(x,\varepsilon) = B(\{x\},\varepsilon)$

$N(A,\varepsilon) = \{x \in X : d(x,A) \leq \varepsilon\}$, $A \in \mathcal{P}(X)$, $\varepsilon > 0$

$CB(X) = \{A \subset X : A$ is closed, bounded and nonempty$\}$.

The set CB(X) will be considered as a metric space endowed with the Hausdorff-Pompeiu metric D:

$D(A,B) = \max\{\sup\{d(x,B) : x \in A\}, \sup\{d(y,A) : y \in B\}\}$.

Accordingly with [1] we will say that a mapping $F:X \to \mathcal{P}(Y)$, where X and Y are topological spaces, is <u>lower semicontinuous (l.s.c.) at</u> $x_0 \in X$ if for any open set $U \subset Y$ with $U \cap F(x_0) \neq \phi$, there exists a neighbourhood V of x_0 such that $F(x) \cap U \neq \phi$ for any $x \in V$; it is <u>upper semicontinuous (u.s.c.) at</u> $x_0 \in X$ if for any open set $U \subset Y$ with $F(x_0) \subset U$ there exists a neighbourhood V of x_0 such that $F(x) \subset U$ for any $x \in V$.

From now on ,X will denote a topological space and (Y,d) a metric one.

In the papers [3,4] it is said that there are point-to-set

mappings $F:X \to CB(Y)$ continuous with respect to the metric D which are not u.s.c.; but in the case that the values of F are compact sets, the u.s.c. is a consequence of the continuity with respect to D. The aim of this note is to weaken the condition of compactness on the images of F, the mentioned implication remaining true. It will be given a condition under which the continuity of F with respect to D is equivalent with the l.s.c and u.s.c. of F.

The following assertion is obvious

<u>Proposition 1.</u> For any $A,B \in CB(Y)$ and $\varepsilon > 0$, we have $D(A,B) \leq \varepsilon$ iff $A \subset N(B,\varepsilon)$ and $B \subset N(A,\varepsilon)$.

<u>Definition 1.</u> A set $A \in \mathcal{P}(Y)$ has <u>the property (*)</u> if for any closed set $C \subset Y$ with $A \cap C = \phi$ we have $d(A,C) > 0$.

<u>Remark 1.</u> Any compact set A has the property (*), as it is shown in [2], p.234. Any infinite set in the space Y with card $Y \geq \aleph_0$ endowed with the discrete metric has the property (*), but it is not compact.

<u>Proposition 2.</u> The property (*) for the set A is equivalent with the fact that $\{N(A,\varepsilon) : \varepsilon > 0\}$ is a base of neighbourhoods for A.

<u>Proof.</u> Let A a set having the property (*) and $U \supset A$ an open set. Then $CU = Y \setminus U$ is closed and $CU \cap A = \phi$, hence $d(A,CU) = \varepsilon > 0$. For $\varepsilon_1 < \varepsilon$ we have $N(A,\varepsilon_1) \subset U$ and then $\{N(A,\varepsilon) : \varepsilon > 0\}$ is a base of neighbourhoods for A. Conversely, let C an arbitrary closed set with $A \cap C = \phi$. Then $A \subset CC$, CC being an open set ; let $\varepsilon > 0$ be such that $A \subset N(A,\varepsilon) \subset CC$. It follows that $d(A,C) \geq d(A,CN(A,\varepsilon)) \geq \varepsilon > 0$.

<u>Theorem 1.</u> For the point-to-set mapping $F:X \to CB(Y)$ the following assertions are true

(a) F is l.s.c. at x_0 if for any $\varepsilon > 0$ there exists a neighbourhood V of x_0 such that $F(x_0) \subset N(F(x),\varepsilon)$, whatever $x \in V$; if $F(x_0)$ is totally bounded the converse is also true.

(b) F is u.s.c. at x_0 implies that for any $\varepsilon > 0$ there exists a

neighbourhood V of x_0 such that $F(x)\subset N(F(x_0),\varepsilon)$, whatever $x\in V$; if $F(x_0)$ has the property (*), the converse is also true.

Proof. (a) Let $x_0\in X$ and F be given such that for any $\varepsilon>0$ there exists V with $F(x_0)\subset N(F(x),\varepsilon)$ whatever $x\in V$. We show that f is l.s.c. at x_0. Let $G\subset Y$ be an open set such that $F(x_0)\cap G\neq\phi$; if $G=Y$, the condition in the definition of l.s.c. is obviously satisfied. Let $G\neq Y$ and $y_0\in F(x_0)\cap G$. We denote $r=d(y_0,\complement G)$. We have $r>0$, because $r=0$ implies $y_0\in\overline{\complement G}=\complement G$, while $y_0\in G$. The hypothesis guarantees the existence of V such that $F(x_0)\subset N(F(x),r/2)$ for any $x\in V$. We prove now that for any $x\in V$, $F(x)\cap G\neq\phi$. Let us suppose that there exists $x\in V$ such that $F(x)\cap G=\phi$, hence $F(x)\subset\complement G$. We obtain then $r=d(y_0,\complement G)\leq d(y_0,F(x))\leq r/2$, since $y_0\in F(x_0)$ and $F(x_0)\subset N(F(x),r/2)$ for $x\in V$. This contradiction shows that $F(x)\cap G\neq\phi$ for any $x\in V$, therefore F is l.s.c. at x_0.

For the converse we suppose $F(x_0)$ totally bounded and F l.s.c. at x_0. Let $\varepsilon>0$ be given and $F(x_0)\subset\bigcup_{k=1}^{n}B(y_k,\varepsilon/2)$, such that $B(y_k,\varepsilon/2)\cap F(x_0)\neq\phi$, $k=\overline{1,n}$. We can then find neighbourhoods V_k of x such that $F(x)\cap B(y_k,\varepsilon/2)\neq\phi$ for any $x\in V_k$, $k=\overline{1,n}$. Let $V=\bigcap_{k=1}^{n}V_k$ and x be in V. For every $y\in F(x_0)$, we obtain from the total boundedness of $F(x_0)$ a point y_k such that $d(y,y_k)<\varepsilon/2$. Because $F(x)\cap B(y_k,\varepsilon/2)\neq\phi$ for any $x\in V$, we obtain $z_k\in F(x)$ such that $d(y_k,z_k)<\varepsilon/2$. It follows that $d(y,z_k)<\varepsilon$, hence $y\in B(z_k,\varepsilon)\subset N(F(x),\varepsilon)$. In conclusion, for any $x\in V$ we have $F(x_0)\subset N(F(x),\varepsilon)$.

(b) Let F be u.s.c. at x_0. For $\varepsilon>0$, we have $F(x_0)\subset B(F(x_0),\varepsilon)$ which is an open set. It follows that there exists a neighbourhood V of x_0 such that $F(x)\subset B(F(x_0),\varepsilon)\subset N(F(x_0),\varepsilon)$ for any $x\in V$.

Conversely, let $F(x_0)$ satisfy the condition (*), hence by Prop.2 $\{N(F(x_0),\varepsilon)\,;\,\varepsilon>0\}$ is a base of neighbourhoods for $F(x_0)$. Let $G\subset Y$ be an open set such that $F(x_0)\subset G$. We obtain an $\varepsilon>0$ such that $F(x_0)\subset N(F(x_0),\varepsilon)\subset G$. By the hypothesis, for this ε we have a neighbourhood V of x_0 such that $F(x)\subset N(F(x_0),\varepsilon)$; for $x\in V$ we have $F(x)\subset G$, hence

F is u.s.c. at x_0.

Theorem 2. In the conditions of Th.1, we have:

(a) If $F(x_0)$ is totally bounded and F u.s.c. and l.s.c. at x_0, then F is continuous at x_0 with respect to the metric D.

(b) If $F(x_0)$ satisfies (*) and F is continuous at x_0 with respect to the metric D, then F is u.s.c. and l.s.c. at x_0.

Proof. (a) For $\varepsilon>0$, we obtain from Th.1 the existence of the neighbourhoods V_1 and V_2 of x_0 such that $F(x)\subset N(F(x_0),\varepsilon)$ for any $x\in V_1$ and $F(x_0)\subset N(F(x),\varepsilon)$ for any $x\in V_2$. Considering $V=V_1\cap V_2$, we have by Prop.1 that $D(F(x),F(x_0))\leq\varepsilon$ for any $x\in V$, i.e. F is continuous at x_0 with respect to D.

(b) For $\varepsilon>0$, the continuity of F with respect to D at x_0 implies the existence of V such that $F(x)\subset N(F(x_0),\varepsilon)$ and $F(x_0)\subset N(F(x),\varepsilon)$ for any $x\in V$. Applying Th.1, we have finally that F is u.s.c. and l.s.c. at x_0.

Remark 2. In the metric space (Y,d), with $Y=(0,1]$ and d the usual metric, $A=(0,1/2]$ is a closed, totally bounded set which satisfies (*), but it is not compact. The point-to-set mapping $F:Y\to CB(Y)$ given by $F(x)=(0,x]$ satisfies the conditions of Th.2 at $1/2$, $F(1/2)$ being not compact.

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