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A SURJECTIVITY THEOREM FOR LIPSCHITZ MAPPINGS

by

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The main result of this paper is a theorem which generalizes the results in the book of V.A.Trenogin /2/ on continuation method (initiated by S.N.Bernstein in 1904) applied to linear operators and also a theorem of F.E.Browder /1/ on Lipschitz mappings. The basic idea is to obtain the bijectivity of the linear operator (Lipschitz mapping) T_1 , knowing that T_0 is bijective and imposing a condition on the segment $T_s = (1-s)T_0 + sT_1$ or on a more general function which replaces this segment.

THEOREM. Let (X,d) be a metric space, $(Y,\|\cdot\|)$ a Banach space and $T : [0,1] \longrightarrow Lip(X,Y)$ a continuous function for which we denote $T(s) = T_s$, $s \in [0,1]$. If for each $s \in [0,1]$ there exists $c_s > 0$ such that

(1) $\quad d(u,v) \leq c_s \| T_s u - T_s v\|$ for each u,v in X

and T_0 is surjective, then T_1 is surjective too.

Remark 1. The mappings considered in the theorem satisfying the condition (1), the surjectivity means in fact bijectivity. In the linear space $Lip(X,Y)$ of the Lipschitz mappings $L : X \longrightarrow Y$ one considers the seminorm

$$\| L \|_{Lip(X,Y)} = \sup\left\{ \frac{\|Lu-Lv\|}{d(u,v)} : u,v \in X,\ u \neq v \right\}.$$

The continuity of T in the theorem means that $s_n \longrightarrow s$ implies $\|T(s_n) - T(s)\|_{Lip(X,Y)} \longrightarrow 0$ for each convergent sequence $(s_n)_{n\in\mathbb{N}}$.

Remark 2. In the hypotheses of the theorem, one may consider in the condition (1) the same constant for each s, without loss of generality. So in the theorem we may impose

(2) there exists $c>0$ such that $d(u,v) \leq c\,\|T_s u - T_s v\|$ for each s in $[0,1]$ and u,v in X.

This fact will be proved in a similar manner as Browder proved it in the case when $T_s = (1-s)T_0 + sT_1$.

Let s in $[0,1]$ be given and c_s the smallest constant for which (1) takes place. We prove that $s \mapsto c_s$ is a upper semicontinuous function, hence upper bounded.

Let s be arbitrary but fixed in $[0,1]$ and $t \in [0,1]$. We have

$$d(u,v) \leq c_s\|T_s u - T_s v\| = c_s\|T_t u - T_t v - T_t u + T_s u - T_s v + T_t v\| \leq c_s\|T_t u - T_t v\| + c_s\|(T_s - T_t)u - (T_s - T_t)v\| \leq$$
$$\leq c_s\|T_t u - T_t v\| + c_s\|T_s - T_t\|_{Lip(X,Y)}\cdot d(u,v).$$

It follows from the continuity of Tthat for t in $[0,1]$ such that $|s-t|$ is sufficiently small we have $c_s\|T_s - T_t\|_{Lip(X,Y)} < 1$. For such t in $[0,1]$ we obtain

$$(1 - c_s\|T_s - T_t\|_{Lip(X,Y)})d(u,v) \leq c_s\|T_t u - T_t v\| \quad \text{and}$$

$$d(u,v) \leq c_s(1 - c_s\|T_s - T_t\|_{Lip(X,Y)})^{-1}\cdot\|T_t u - T_t v\|.$$

But c_t is the smallest constant for which such arelation holds, so we obtain

$$c_t \leq c_s(1 - c_s\|T_s - T_t\|_{Lip(X,Y)})^{-1}$$

and the upper semicontinuity of $s \mapsto c_s$ follows.

The proof of the theorem. The Remark 2 allows us to make the proof using condition (2) instead of (1).

Let $H = \{s \in [0,1] : T_s \text{ is surjective}\}$. Of course H is nonvoid ($0 \in H$) and $H \subseteq [0,1]$. We prove that H is both open and closed in $[0,1]$, which means that $H = [0,1]$.

We shall use the fact that (2) implies

(3) $d(T_s^{-1}y, T_s^{-1}z) \leq c\|y-z\|$ for each s in H, y,z in Y

and also

(4) $\|T_s^{-1}\|_{Lip(Y,X)} := \sup\left\{d(T_s^{-1}y, T_s^{-1}z)/\|y-z\| : y,z \in Y, y \neq z\right\} \leq c$ for each s in H.

a) H is open in $[0,1]$.

Let t be given in H. We have $\|(T_s - T_t)\,T_t^{-1}\|_{Lip(Y,Y)} \leq \|T_s - T_t\|_{Lip(X,Y)} \cdot \|T_t^{-1}\|_{Lip(Y,X)} \leq c\,\|T_s - T_t\|_{Lip(X,Y)} < 1$ for $|s-t|$ sufficiently small.

Hence $(T_s - T_t)\,T_t^{-1}$ is a contraction, so $1_Y + (T_s - T_t)\,T_t^{-1} = T_s\,T_t^{-1}$ is surjective. It follows the surjectivity of T_s for $|s-t|$ sufficiently small, hence H is open.

b) H is closed in $[0,1]$.

Let $s \in \overline{H}$, so there exists a sequence $s_n \in H$, $s_n \to s$. We obtain similarly $\|(T_{s_n} - T_s)\,T_{s_n}^{-1}\|_{Lip(Y,Y)} \leq c\,\|T_{s_n} - T_s\|_{Lip(X,Y)} < 1$ for $n \geq n_0$ and the surjectivity of T_s follows as above.

We obtain $H = [0,1]$, hence T_1 is also surjective.

The results of Browder and Trenogin are consequences of this theorem.

Corollary 1. For $T : [0,1] \longrightarrow Lip(X,Y)$, $T_t = (1-t)T_0 + tT_1$ one obtains Theorem 7 in /1/.

Proof. We have to check only that $T : [0,1] \longrightarrow Lip(X,Y)$ is continuous. But $\|T_s - T_t\|_{Lip(X,Y)} = |s-t|\cdot\|T_1 - T_0\|_{Lip(X,Y)}$ and the continuity follows.

Corollary 2 /2/. Let X, Y be Banach spaces, $T : [o,1] \longrightarrow (X,Y)^*$ a continuous function, $(X,Y)^*$ denoting the set of continuous linear operators $L : X \longrightarrow Y$. If T_o is bijective and there exists $\gamma > o$ such that

(5) $\| T_s x \| \geq \gamma \| x \|$ for each s in $[o,1]$ and x in X

then T_1 is bijective too and $\| T_1^{-1} \| \leq 1/\gamma$.

Proof. It is obvious that $(X,Y)^* \subseteq Lip(X,Y)$ and $\| T \|_{Lip(X,Y)} = \| T \|$, hence $T : [o,1] \longrightarrow Lip(X,Y)$ is also continuous. Using the condition (5) we obtain

$$\| u-v \| \leq \gamma^{-1} \| T_s(u-v) \| = \gamma^{-1} \| T_s u - T_s v \| ,$$

hence (2) takes place with $c = \gamma^{-1}$ and the theorem applies.

In the following we apply the theorem to obtain existence theorems for a class of nonlinear boundary value problems. In fact, the function T in the example is a "segment", so Corollary 1 applies. A similar example is given in /2/ in the linear case.

We consider the boundary value problem

$$\begin{cases} -x''(t) + a(t)x'(t) + b(t)g(x(t)) = y(t), \ t \in [o,1] \\ x(o) = x(1) = o \end{cases}$$

where $a : [o,1] \to [-A,A]$, $b : [o,1] \to [\beta,B]$, $A \geq o$, $\beta > o$, $g : R \to R$ are continuous functions such that

$$o < L \leq (g(x_1) - g(x_2))/(x_1 - x_2) \leq K$$

for each $x_1, x_2 \in R$, $x_1 \neq x_2$.

Let $X = \{ x \in C^2[o,1] : x(o) = x(1) = o \}$, $\| x \|_X = \| x \| + \| x' \| + \| x'' \|$ and $Y = C[o,1]$ with $\| \cdot \|$ the sup norm.

Consider $s \in [o,1]$ and

$$(T(s))(x) = -x''(.) + s(a(.)x'(.) + b(.)g(x(.))).$$

We have to show that

a) $T(s) = T_s \in Lip(X,Y)$

b) $\| x_1 - x_2 \|_X \leq c_s \| T_s x_1 - T_s x_2 \|$.

$T : [o,1] \to Lip(X,Y)$ being continuous as it was shown in the proof of Corollary 1.

a) We estimate

$$\begin{aligned} | T_s x_1(t) - T_s x_2(t) | &\leq \| x_1'' - x_2'' \| + sA \| x_1' - x_2' \| + \\ &+ sB | g(x_1(t)) - g(x_2(t)) | \leq \\ &\leq \| x_1'' - x_2'' \| + sA \| x_1' - x_2' \| + sBK \| x_1 - x_2 \| \leq \\ &\leq K_s \| x_1 - x_2 \|_X , \end{aligned}$$

hence

$$\frac{\| T_s x_1 - T_s x_2 \|}{\| x_1 - x_2 \|_X} \leq K_s, \quad \forall \ x_1 \neq x_2,$$

where $K_s = \max \{ 1, sA, sBK \}$.

b) For $s = o$ we have

$y := T_o x = -x''$ and

$$x(t) = \int_o^t E(u,t) y(u) du, \quad E(u,t) = \begin{cases} u(1-t), & o \leq u \leq t \\ t(1-u), & t < u \leq 1 \end{cases},$$

hence

$$\| x_1 - x_2 \| \leq \| y_1 - y_2 \| .$$

It is obvious that

$$\| x_1'' - x_2'' \| = \| y_1 - y_2 \| .$$

Because of $(x_1 - x_2)(o) = (x_1 - x_2)(1)$ there exists $\xi \in (o,1)$ such that $(x_1' - x_2')(\xi) = o$, hence by integrating

$$(x_1'' - x_2'')(t) = -(y_1 - y_2)(t)$$

one has

$$(x_1' - x_2')(t) = -\int_\xi^t (y_1 - y_2)(u) du, \text{ and}$$

$$\| x_1' - x_2' \| \leq \| y_1 - y_2 \| .$$

It follows

$$\| x_1 - x_2 \|_X \leq 3 \| T_o x_1 - T_o x_2 \| .$$

Now let $s \neq 0$ and denote $y_i := T_s x_i$, $i = \overline{1,2}$.

We obtain

$$|x_1''(t) - x_2''(t)| = |sa(t)x_1'(t) + sb(t)g(x_1(t)) - y_1(t) - sa(t)x_2'(t) - sb(t)g(x_2(t)) + y_2(t)| \leq$$
$$\leq sA\|x_1' - x_2'\| + sBK\|x_1 - x_2\| + \|y_1 - y_2\|, \text{ so}$$

$$\|x_1'' - x_2''\| \leq sA\|x_1' - x_2'\| + sBK\|x_1 - x_2\| + \|y_1 - y_2\| . \tag{5}$$

Let $r \in (0,1)$ such that $x_1'(r) - x_2'(r) = 0$.

We multiply

$$-x_i''(t) + sa(t)x_i'(t) + sb(t)g(x_i(t)) = y_i(t)$$

by $\exp(-s\int_0^t a(u)du)$ and obtain

$$-\left[(x_1'(t) - x_2'(t))\exp(-s\int_0^t a(u)du)\right]' +$$
$$+ sb(t)\left[g(x_1(t)) - g(x_2(t))\right]\exp(-s\int_0^t a(u)du) =$$
$$= \left[y_1(t) - y_2(t)\right]\exp(-s\int_0^t a(u)du).$$

We have

$$x_1'(t) - x_2'(t) = s\int_r^t b(u)\left[g(x_1(u)) - g(x_2(u))\right]\exp(s\int_u^t a(v)dv)du -$$
$$- \int_r^t \left[y_1(u) - y_2(u)\right]\exp(s\int_u^t a(v)dv)du.$$

It follows

$$|x_1'(t) - x_2'(t)| \leq sBKE\|x_1 - x_2\| + E\|y_1 - y_2\| ,$$

where $E = \sup\{\exp(s\int_u^t a(v)dv) : s, u, t \in [0,1]\}$, hence

$$\|x_1' - x_2'\| \leq sBKE\|x_1 - x_2\| + E\|y_1 - y_2\| . \tag{6}$$

Consider now $t_0 \in (0,1)$ such that

$$\|x_1 - x_2\| = |x_1(t_0) - x_2(t_0)| .$$

We may suppose $x_1(t_0) \geq x_2(t_0)$ (otherwise we interchange x_1 and x_2). It follows that t_0 is a maximum point, therefore

$x_1'(t_0) - x_2'(t_0) = 0$ and $x_1''(t_0) - x_2''(t_0) \leq 0$.

We have

$$|x_1(t_0) - x_2(t_0)| = x_1(t_0) - x_2(t_0) \leq$$
$$\leq (g(x_1(t_0)) - g(x_2(t_0)))/L =$$
$$= \frac{1}{b(t)sL}(y_1(t_0) - y_2(t_0) + x_1''(t_0) - x_2''(t_0)) \leq$$
$$\leq \frac{1}{s\beta L}(y_1(t_0) - y_2(t_0)) \leq \frac{1}{s\beta L}\|y_1 - y_2\| , \text{ hence}$$

$$\|x_1 - x_2\| \leq \frac{1}{s\beta L}\|y_1 - y_2\| . \tag{7}$$

Using the conditions (5), (6) and (7) we get

$$\|x_1 - x_2\|_X \leq (sA + 1)\|x_1' - x_2'\| + (sBK + 1)\|x_1 - x_2\| + \|y_1 - y_2\| \leq$$
$$\leq \left[(sA + 1)sBKE + (sBK + 1)\right]\|x_1 - x_2\| +$$
$$+ \left[E(sA + 1) + 1\right]\cdot\|y_1 - y_2\| \leq$$
$$\leq \left[(sA + 1)\frac{BK}{\beta L}E + \frac{BK}{\beta L} + \frac{1}{s\beta L} + E(sA + 1) + 1\right]\|y_1 - y_2\| =$$
$$= \left[\frac{1}{s\beta L} + (1 + E + sAE)\left(\frac{BK}{\beta L} + 1\right)\right]\|y_1 - y_2\| =$$
$$= c_s\|y_1 - y_2\| .$$

It follows that all the conditions in Corollary 1 are satisfied, and $T(1)$ is bijective. This implies that under the given conditions the considered boundary value problem has a unique solution in X for each y in $C[0,1]$.

The continuation method allows us to obtain existence and uniqueness results for some problems which are related by means of a suitable mapping T to a problem for which the existence and uniqueness of the solution is known.

R E F E R E N C E S

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